Math 3201 **Unit 8: SINUSODIAL FUNCTIONS**

<u>Section 8.1: Understanding Angles</u> p. 484

How can we measure things?

Examples:

- Length - meters (m) or yards (yd.)
- Temperature - degrees Celsius (°C) or Fahrenheit (F)

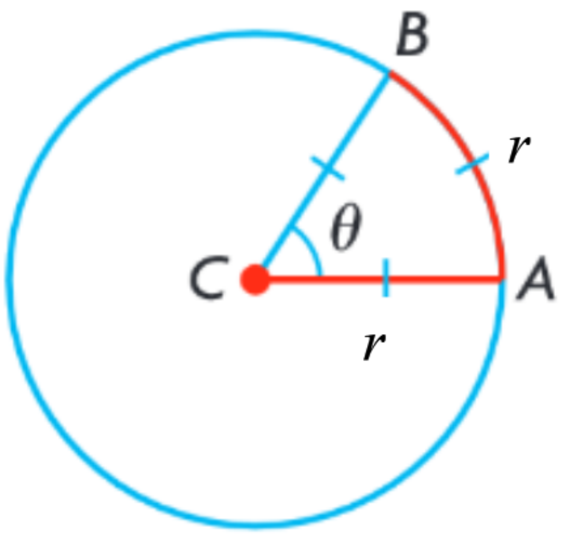
How can we measure angles?

Up until now we can measure angles using "degrees".

There is an alternative unit of measurement for measuring angles, that is, "RADIANS".

RADIAN:

- One radian is the angle made by taking the radius and wrapping it along the edge (an arc) of the circle.



1 radian ~ 60° Degrees

INVESTIGATE:

How many pieces of this length do you think it would take to represent one complete circumference of the circle?

To help, cut a piece of pipe cleaner/string to a length equal to radius CA and bend it around the circle starting at point A.

1. Approximately, how many radius lengths are there in one complete circumference? 6

2. How many degrees in one complete circumference? 360°

3. Therefore, approximately, how degrees are in one radian? $\frac{360}{6} = 60°$

NOTE: 1. The size of the radius of a circle has NO effect on the size of 1 radian.

2. The advantage of radians is that it is directly related to the radius of the circle. This means that the units of the x and y axis is consistent and the graph of the sine curve will have its true shape, without vertical exaggeration.

r = radius d = diameter

Let's consider a circle with radius 1 unit ($r = 1$). (We refer to this as the ***unit circle***!)

1. What is the circumference of a circle? $C = 2\pi r$ or $C = \pi d$
2. What is the circumference of the *unit* circle? $C = 2\pi(1) = 2\pi$
3. How many degrees are there in a complete revolution of a circle? $360°$
4. Why must the two equations be equal to each other? $2\pi = 360°$
   Because they are the same!
5. State the proportion: $2\pi : 360$

6. Divide both sides by 2π radians to determine the measure of 1 radian.
   $\frac{2\pi}{2\pi} : \frac{360}{2\pi} \rightarrow 1 : \frac{180}{\pi} \rightarrow \boxed{1 \text{ radian} = \frac{180}{\pi}}$

7. Complete the following and add the equivalent radian measures on the unit circle below:

   a. $2\pi =$ $360°$

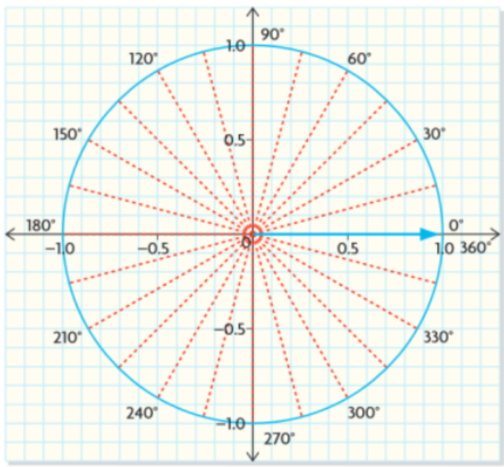
   b. $\pi =$ $180°$

   c. $\frac{\pi}{2} =$ $90°$

   d. $\frac{\pi}{4} =$ $45°$

   e. $\frac{\pi}{6} =$ $30°$

   f. $\frac{\pi}{3} =$ $60°$



$\frac{\cancel{\pi}}{6} \cdot \frac{180}{\cancel{\pi}} = \frac{180}{6} = 30°$ | $\frac{\cancel{\pi}}{3} \cdot \frac{180}{\cancel{\pi}} = \frac{180}{3} = 60°$ | $\cancel{\pi} \cdot \frac{180}{\cancel{\pi}} = 180°$

If given radians, multiply by $\frac{180}{\pi}$ to convert to degrees!

Converting Degrees ⟺ Radians $\div \frac{180}{\pi} \Rightarrow \times \frac{\pi}{180}$

1. Degrees to Radians:
   - To convert from Degrees to Radians multiply by $\frac{\pi}{180^o}$
2. Radians to Degrees:
   - To convert from Radians to Degrees multiply by $\frac{180^o}{\pi}$

$\frac{120}{180}$ $\quad \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$

Examples:

1. Convert to radians:

   a. 120° $\quad 120 \times \frac{\pi}{180} = \frac{120\pi}{180}$ (÷60) $= \boxed{\frac{2\pi}{3}}$

   b. 240° $\quad 240 \times \frac{\pi}{180} = \frac{240\pi}{180}$ (÷60) $= \boxed{\frac{4\pi}{3}}$

   c. 450° $\quad 450 \times \frac{\pi}{180} = \frac{450\pi}{180}$ (÷90) $= \boxed{\frac{5\pi}{2}}$

   d. 660° $\quad 660 \times \frac{\pi}{180} = \frac{660\pi}{180} = \boxed{\frac{11\pi}{3}}$

2. Convert to degrees:

   a. $\frac{3\pi}{4}$ $\quad \frac{3\cancel{\pi}}{4} \times \frac{180}{\cancel{\pi}}$ $\quad \frac{3 \cdot 180}{4} = \frac{540}{4} = \boxed{135^o}$

   b. $\frac{13\pi}{6}$ $\quad \frac{13\cancel{\pi}}{6} \times \frac{180}{\cancel{\pi}}$ $\quad \frac{13 \cdot 180}{6} = 390^o$

   c. $\frac{4\pi}{9}$ $\quad \frac{4\cancel{\pi}}{9} \cdot \frac{180}{\cancel{\pi}}$ $\quad \frac{4 \cdot 180}{9} = 80^o$

   d. 5.3 rad $\quad 5.3 \cdot \frac{180}{\pi} = \frac{5.3(180)}{\pi} = 303.7^o$ $\quad (\pi \approx 3.14)$

3. For each pair of angle measures, determine which is greater:

   a. 150°, 2 $\quad 2 \times \frac{180}{\pi} = 114.6^o < \boxed{150^o}$

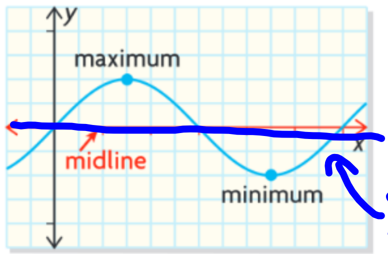
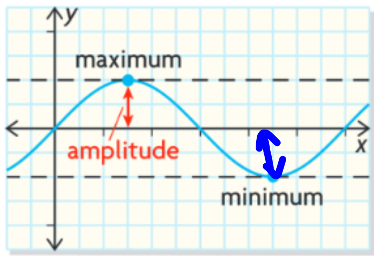
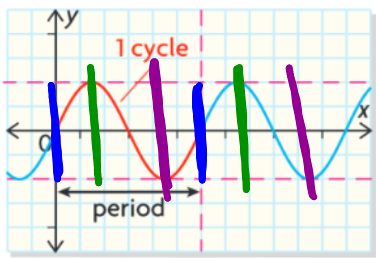
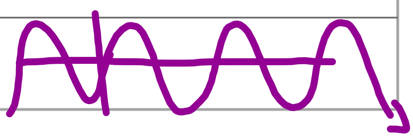
   b. 450°, 7 $\quad 7 \times \frac{180}{\pi} = 401.1^o < \boxed{450^o}$

   c. 3π, 8 $\quad 3\pi = 3 \times \pi = 9.42$ rad

ASSIGN: p. 489, #1, 2, 5, 7 - 10

<u>Section 8.2: Exploring Graphs of Periodic Functions</u> p. 491

Terms to Know:

Periodic Function	A function whose graph repeats in regular intervals or cycles.	
Midline / Sinusodial Axis SA	The horizontal line halfway between the maximum and minimum values of a periodic function.	 SA
Amplitude	The distance from the midline to either the maximum or minimum value of a periodic function; the amplitude is always expressed as a positive number.	
Period	The length of the interval of the domain to complete one cycle.	
Sinusodial Function	Any periodic function whose graph has the same shape as that of $y = \sin x$.	

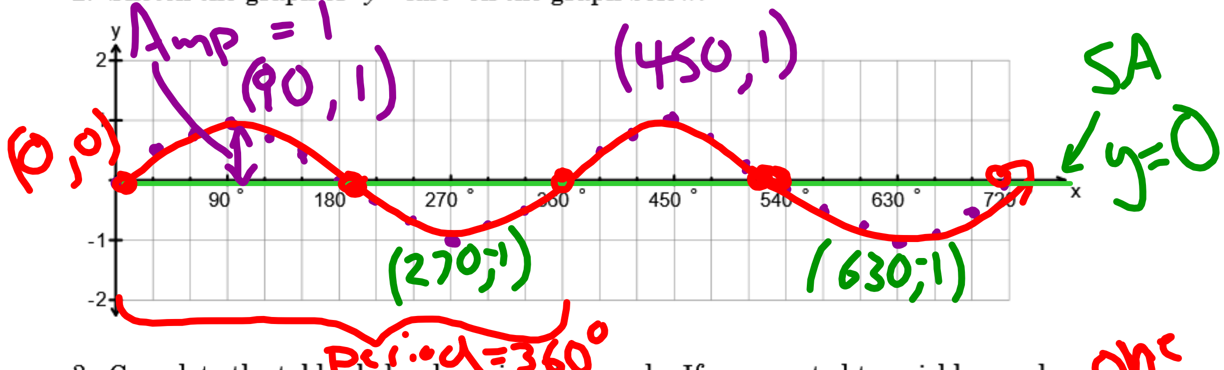
Section 8.2: EXPLORATION – The Sine Curve

1. Complete the table of values below for the function $y = \sin\theta$

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	225°	270°	315°	360°
y	0	0.5	0.707	0.866	1	0.866	0.707	0.5	0	-0.707	-1	-0.707	0

θ	390°	420°	450°	480°	510°	540°	585°	630°	675°	720°
y	0.5	0.866	1	0.866	0.5	0	-0.707	-1	-0.707	0

2. Sketch the graph of $y = \sin\theta$ on the graph below:



3. Complete the tables below by using the graph. If you wanted to quickly graph the sine curve, which five points would allow you to easily graph the entire curve?

one cycle

"Five Key Points"

	$y = \sin\theta$
Period	360°
Sinusoidal Axis (midline)	y=0
Amplitude	1
Domain	$x \in \mathbb{R}$
Range	$\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$
Local Maximums	(90,1), (450,1) ...
Local Minimums	(270,-1), (630,-1)
x-intercepts	0, 180, 360, 540, 720
y-intercepts	(0,0)

x	y
0°	0
90°	1
180°	0
270°	-1
360°	0

Section 8.2: EXPLORATION – The Cosine Curve

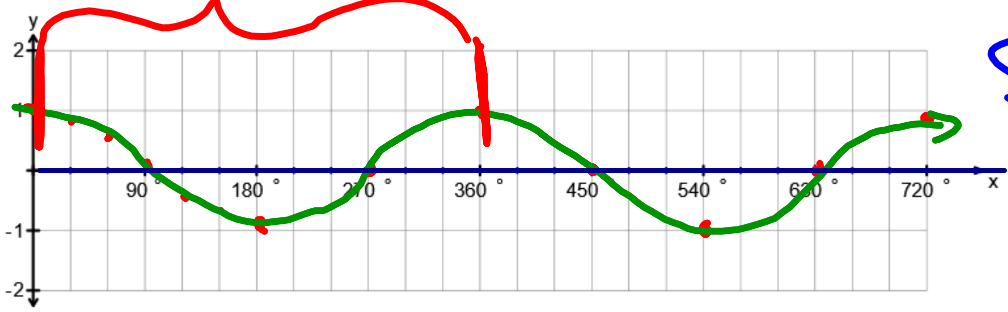
1. Complete the table of values below for the function $y = \cos\theta$

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	225°	270°	315°	360°
y	1	0.866	0.707	0.5	0	-0.5	-0.707	-0.866	-1		0		1

θ	390°	420°	450°	480°	510°	540°	585°	630°	675°	720°
y			0			-1		0		1

Period = 360

2. Sketch the graph of $y = \cos\theta$ on the graph below:



SA
y=0

3. Complete the tables below by using the graph. If you wanted to quickly graph the cosine curve, which five points would allow you to easily graph the entire curve?

	$y = \cos\theta$
Period	360°
Sinusoidal Axis (midline)	y=0
Amplitude	1
Domain	$x \in \mathbb{R}$
Range	$\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$
Local Maximums	(0,1), (360,1), (720,1) ...
Local Minimums	(180,-1), (540,-1)
x-intercepts	90, 270, 450 ...
y-intercepts	(0,1)

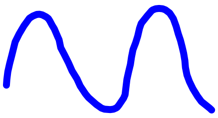
"Five Key Points"

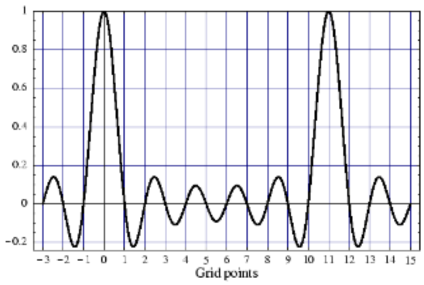
x	y
0°	1
90°	0
180°	-1
270°	0
360°	1

OBSERVATIONS:

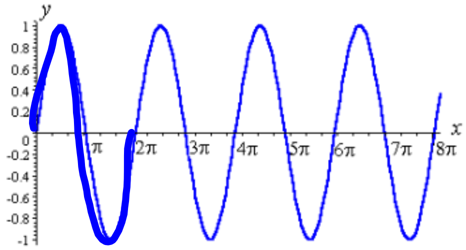
- For SINE, one complete wave can be seen from an X-INTERCEPT at (0, 0) to the X-INTERCEPT at (360°, 0).
- For COSINE, one complete wave can be seen from the MAXIMUM point (0, 1) to the next MAXIMUM point at (360°, 1).
- The graph of $y = \cos\theta$ is related to the graph of $y = \sin\theta$ by a **shift of 90° to the left**.

Label the following as periodic, sinusoidal or both.

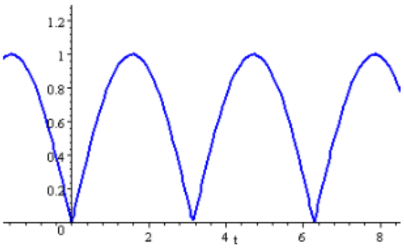




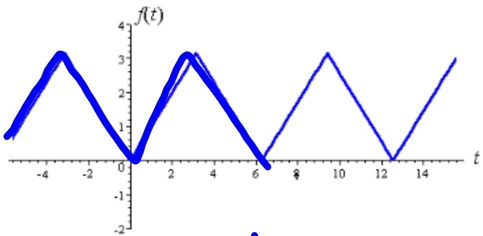
a. Periodic



c. Both



b. Periodic



d. Periodic

CONCLUSION:

All sinusoidal functions are periodic but not all periodic functions are sinusoidal.

ASSIGN: p. 494, #2, 5 - 8

Section 8.3: The Graphs of Sinusoidal Functions p. 497

Examples:

1. For the sinusoidal function shown, determine: (Example 1, p. 499)

a. Range: Set notation → $\{y \mid -3 \le y \le 7, y \in \mathbb{R}\}$

Single Value: $max - min = 7-(-3) = \boxed{10}$

b. Equation of Midline/ Sinusoidal Axis:

Midline: $\frac{max + min}{2} = \frac{7+(-3)}{2} = \frac{4}{2} = \boxed{2}$ $\boxed{y=2}$

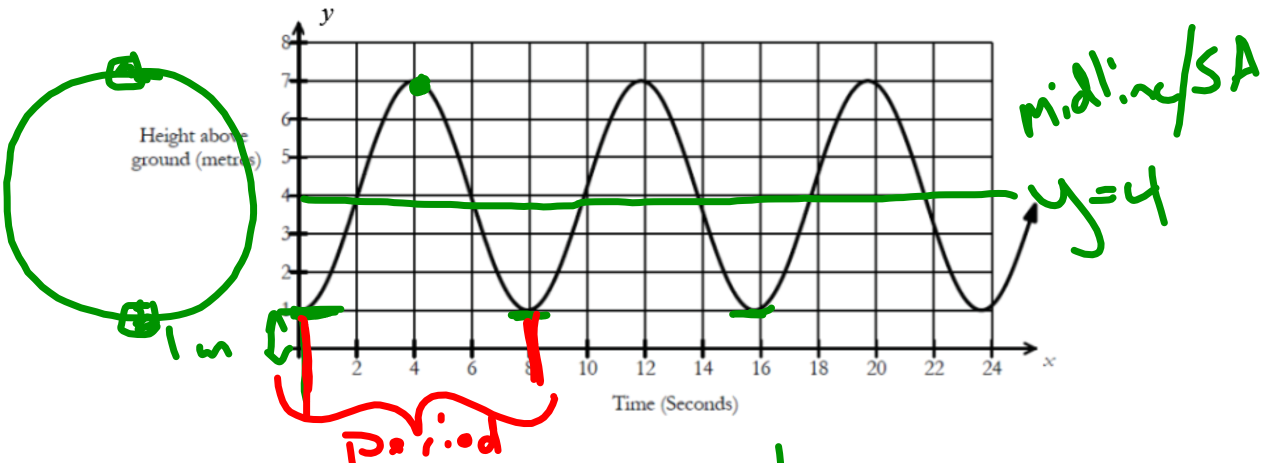
c. Amplitude:

Amp = max − midline $= 7-2 = \boxed{5}$ or $\frac{max - min}{2} = 5$

d. Period:

Period = 180°

2. While riding a Ferris wheel, Mason's height above the ground in terms of time can be represented by the following graph.



a. How far is the ferris wheel off the ground? 1 m

b. What is the range of the function? What does it represent? Max−min → 7−1 = 6

diameter of the ferris wheel

c. What is the height of the ferris wheel? maximum → 7 m

$\frac{7+1}{2} = \frac{8}{2} = 4$

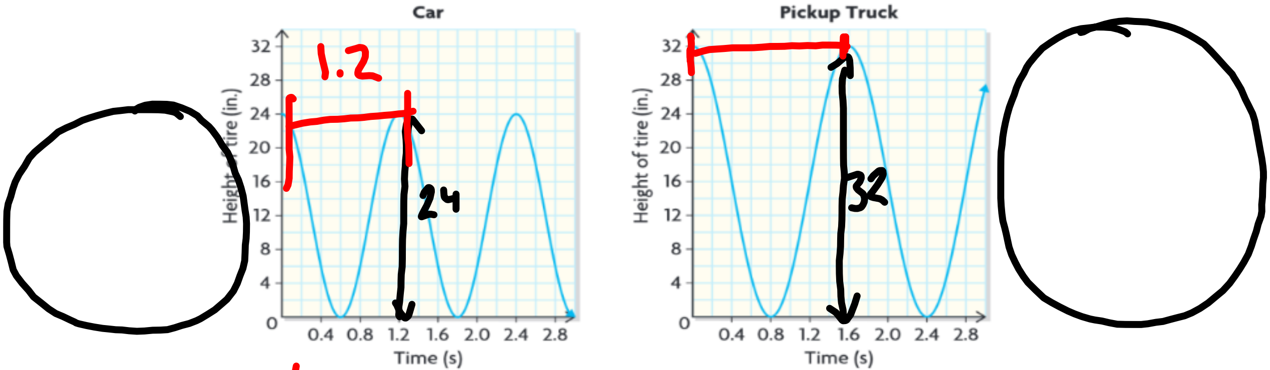
d. What is the equation of the midline? What does it represent?

$y=4$ middle of the wheel

e. How long does it take for the ferris wheel to make one complete revolution? What characteristic does this correspond to? 8 seconds

Period

3. Alexis and Colin own a car and a pickup truck. They noticed that the odometers of the two vehicles gave different values for the same distance. As part of their investigation into the cause, they put a chalk mark on the outer edge of a tire on each vehicle. The following graphs show the height of the tires as they rotated while the vehicles were driven at the same slow, constant speed. What can you determine about the characteristics of the tires from these graphs? (Example 4, p. 504)



Characteristics	Car	Truck
Diameter	24	32
Radius	12	16
Period	1.2 sec	1.6 sec

ASSIGN: p. 507, #4, 5, 7 – 10, 13 - 15

Section 8.4: The Equations of Sinusoidal Functions p. 516

INVESTIGATION:

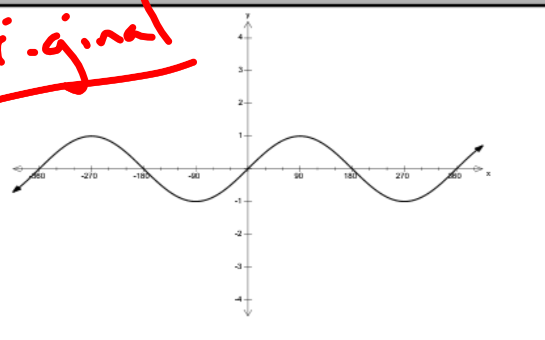
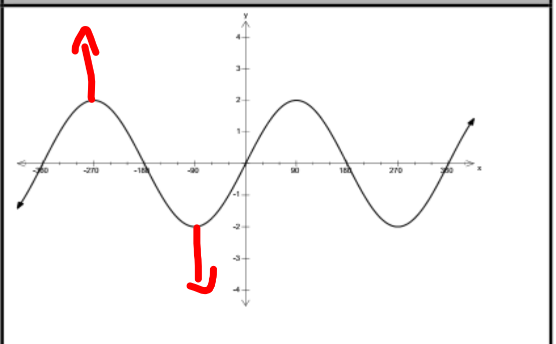
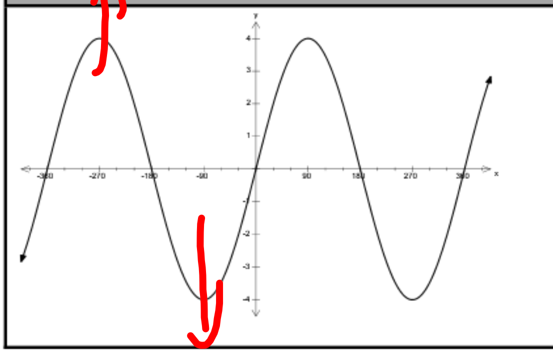
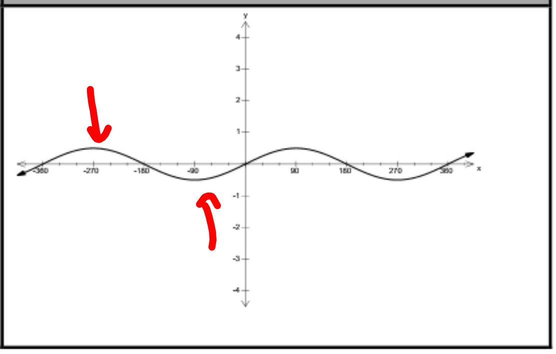
Using technology, we will explore how the parameters a, b, c and d affect the graph of sinusoidal functions written in the form:

$y = a\sin b(x-c)+d$ and $y = a\cos b(x-c)+d$

(A) The Effect of 'a' in $y = a\sin x$ on the graph $y = \sin x$ where $a > 0$

1. Sketch a graph of: $y=\sin x$, $y = 2\sin x$, $y = 4\sin x$, $y = 0.5\sin x$

Original

$y=\sin x$	$y=2\sin x$
	
$y=4\sin x$	$y=0.5\sin x$
	

2. Compare the amplitudes in each graph with its equation.

3. Describe the affect the value of 'a' has on the graph of $y=\sin x$.

"a" changes amplitude a = amplitude

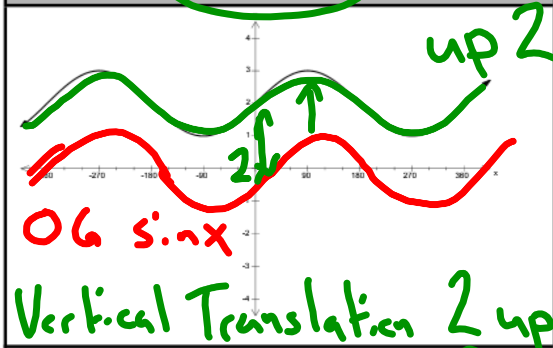
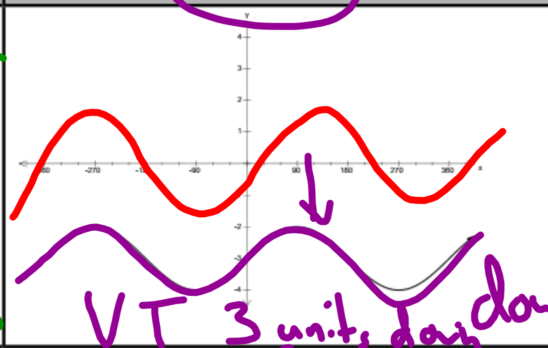
4. Will the value of 'a' affect the cosine graph in the same way that it affects the sine graph?

yes!

a = amplitude (Vertical Stretch)

(B) The Effect of '*d*' in $y = \sin x + d$ on the graph $y = \sin x$

5. Sketch a graph of: $y = \sin x + 2$ and $y = \sin x - 3$ and compare it to the graph of $y = \sin x$

$y = \sin x + 2$	$y = \sin x - 3$
	

6. How does each graph change when compared to $y = \sin x$? ____________

7. How is the value of '*d*' related to the equation of the midline? d = midline

8. Is the ~~shape~~ of the graph or the location of the graph affected by the parameter '*d*'?
location

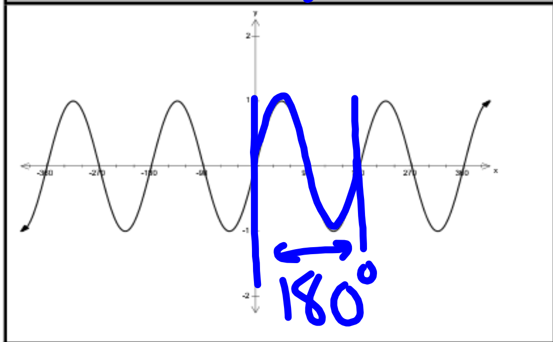
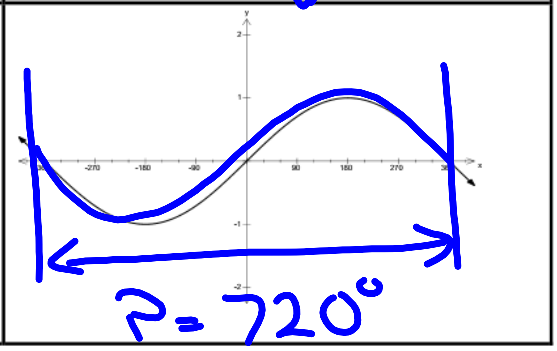
9. Is the period affected by changing the value of '*d*'? nope

10. Will the value of '*d*' affect the cosine graph in the same way that it affects the sine graph?
yup!

"d" = midline (Vertical Translation)

(C) The Effect of 'b' in $y = \sin bx$ on the graph $y = \sin x$

11. Sketch a graph of: $y = \sin 2x$ and $y = \sin 0.5x$ and compare it to the graph of $y = \sin x$

$y = \sin 2x$	$y = \sin 0.5x$
	

12. What is the period of $y = \sin x$? 360° What is the 'b' value? 1 $\sin x = \sin 1x$

13. What is the period of $y = \sin 2x$? 180° What is the 'b' value? 2

14. What is the period of $y = \sin 0.5x$? 720° What is the 'b' value? 0.5 = ½

15. What is affected by the value of 'b'? Period

16. Write an equation that relates the 'b' value to the period of the function.

Degrees → $\text{Period} = \dfrac{360^\circ}{b}$ or $\text{Period} = \dfrac{2\pi}{b}$ → Radians

17. Will the value of 'b' affect the cosine graph in the same way that it affects the sine graph?

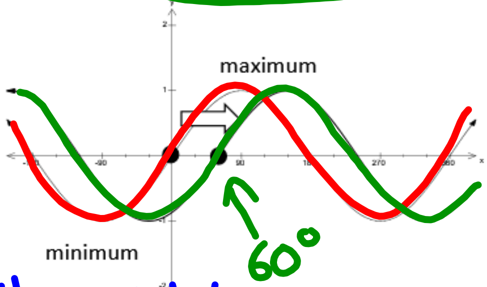
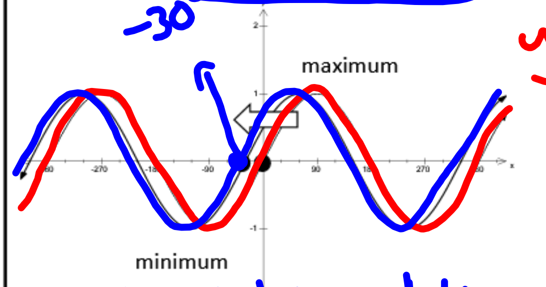
yes

Conclusion: The value of b affects the period. Period = $\dfrac{360^\circ}{b}$ → Horizontal Stretch

(D) The Effect of '*c*' in $y = \sin(x - c)$ on the graph $y = \sin x$

18. Sketch a graph of: $y = \sin(x - 60^o)$ and $y = \sin(x + 30^o)$ and compare it to the graph of $y = \sin x$

y=sin x

$y = \sin(x - \boxed{60^o})$	$y = \sin(x + 30^o) = \sin(x - (-30^o))$
 Horizontal translation of 60° right	-30  y=sin x Horizontal translation of 30° left.

19. How is the '*c*' value affecting the graph? Horizontal translation

This horizontal shift is also called the ***phase shift*** of the graph. In order to determine the phase shift of the graph, you need to compare a **KEY POINT** on the sine graph to determine if it has shifted **left** or **right**.

One key point on the **SINE** graph is the point **(0, 0)** which intersects the midline at $x = 0$ going **from a minimum point to a maximum**.

20. If the '*c*' value is positive, the graph shifts to the right.

21. If the '*c*' value is negative, the graph shifts to the left.

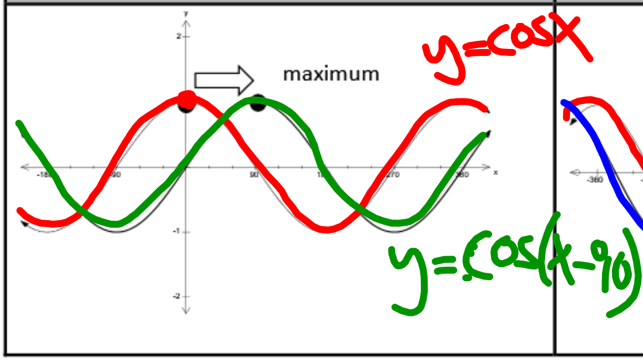
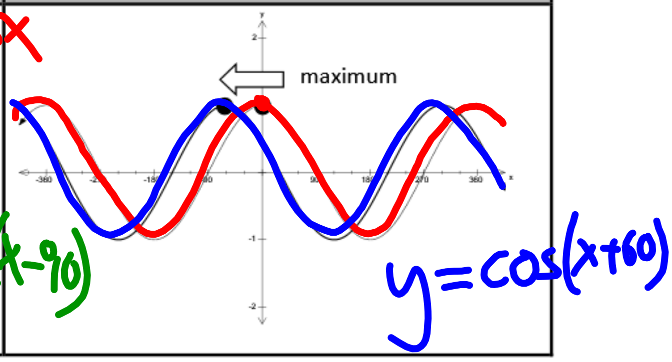
22. Will the value of '*c*' affect the cosine graph in the same way that it affects the sine graph?

Yes! It will be the same translation (horizontal) but a different keypoint.

(E) The Effect of 'c' in $y = \cos(x - c)$ on the graph $y = \cos x$

23. Sketch a graph of: $y = \cos(x - \boxed{90^o})$ and $y = \cos(x + 60^o)$ and compare it to the graph of $y = \sin x$

$= \cos(x - (-60))$

$y = \cos(x - 90^o)$	$y = \cos(x + 60^o) = \cos(x - (-60^0))$
	

One KEY point on the **COSINE** graph is the point **(0, 1)** which is a **maximum point.**

24. If the 'c' value is positive, the graph shifts to the right. ex $y=\cos(x-90)$

25. If the 'c' value is negative, the graph shifts to the left. ex $y=\cos(x+60)$ $=\cos(x-(-60))$

CONCLUSION: For the sinusoidal functions written in the form:

$$y = a \sin b(x - c) + d \quad \text{and} \quad y = a \cos b(x - c) + d$$

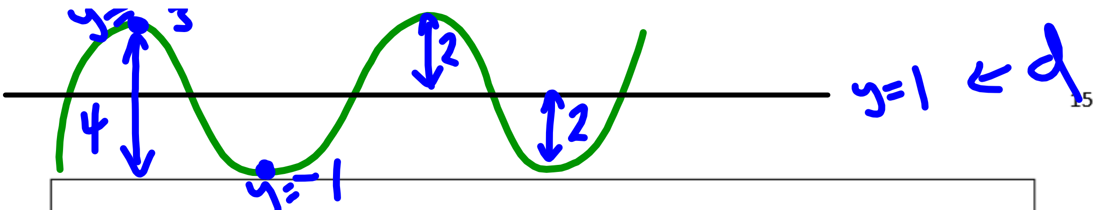
1. a = amplitude = $\dfrac{\max - \min}{2}$

2. b affects the period: $period = \dfrac{360^o}{b}$ or $period = \dfrac{2\pi}{b}$

3. c = horizontal shift (need to compare a Key point)

4. d = vertical translation, $y = d = \dfrac{\max + \min}{2}$ = equation of the midline/sinusoidal axis

5. maximum value $= d + a$; minimum value $= d - a$



NOTE:	Beware of the brackets!	$y = \cos(x+2)$ versus $y = \cos x + 2$
	$y = \cos(x+2)$ results in a horizontal translation	2 → c
	$y = \cos x + 2$ results in a vertical translation	2 → d

$y = a\cos b(x-c) + d$

Examples:

1. For the function, $y = 2\cos 4x + 1$, state: (Example 1, p. 518)

   $a = 2$, $b = 4$, $c = 0$, $d = 1$

   a → a. amplitude: 2

   d → b. equation of the midline: $y = 1$

   c. range: **max:** $d + a = 1 + 2 = 3$ **min:** $d - a = 1 - 2 = -1$ Range = max − min $= 3 - (-1)$ $\boxed{= 4}$

   d. period: **period** $= \frac{360}{b} = \frac{360}{4} = 90°$

   e. phase shift: $c = 0$ → no phase shift (horizontal translation)

2. For the function, $y = 3\sin 2(x - \boxed{45})$, state: (Example 2, p.519)

   $a = 3$, $b = 2$, $c = 45°$, $d = 0$

   a. amplitude: 3

   b. equation of the midline: $y = 0$

   c. range: $max = 0 + 3 = 3$, $min = 0 - 3 = -3$ Range = max − min $= 3 - (-3) = 6$

   d. period: $= \frac{360}{b} = \frac{360}{2} = 180°$

   e. phase shift: $c = 45°$ 45° right

   f. How would the graph of $y = 3\cos 2(x + 45°)$ be the same? How would it be different?

Same	different
amplitude midline period range	phase shift 45° left

$y = a\sin b(x-c)+d$ $\quad$ $y = a\cos b(x-c)+d$

$y = 4\sin 3(x-60^\circ)+1$ $\quad$ $y = 4\cos 3(x-90^\circ)+1$

$a = 4$
$b = 3$
$c =$
$d = 1$

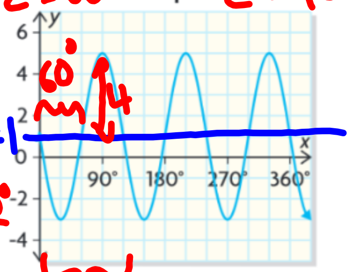
$\text{Period} = \dfrac{360^\circ}{b}$

$120 = \dfrac{360^\circ}{b}$

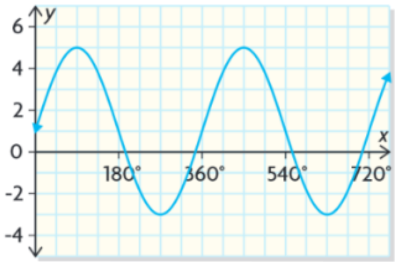
$b = \dfrac{360}{120} = 3$

3. Match each graph with the corresponding equation below. (Example 3, p. 522)

a) sin $c = 60^\circ$ $\quad$ cos $c = 90^\circ$



b)



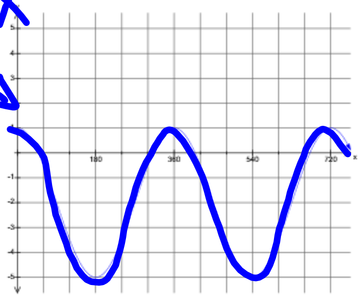
i. $y = 4\cos(x-90^\circ)+1$ $\qquad$ iii. $y = 5\sin 3(x-60^\circ)$

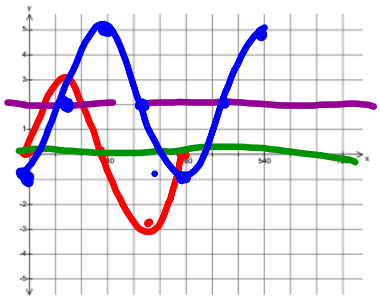
ii. $y = 4\sin 3(x-60^\circ)+1$ $\qquad$ iv. $y = 4\cos 3(x-60^\circ)+1$

4. Ashley created the following graph for the equation $y = 3\sin(x-90^\circ)+2$. Identify her error(s) and construct the correct graph.

midline at $y = -2$ X

90° shift left X





$a = 3$
$b = 1$
$c = 90$
$d = 2$ midline
90° right

5. The temperature of an air-conditioned home on a hot day can be modeled using the function $t(x) = 1.5(\cos 15^\circ x)+20$, where x is the time in minutes after the air conditioner turns on and $t(x)$ is the temperature in degrees Celsius.

$a = 1.5$
$b = 15$
$c = 0$
$d = 20$

a. What are the maximum and minimum temperatures in the home?

max = 20 + 1.5 = 21.5
min = 20 − 1.5 = 18.5

b. What is the temperature 10 minutes after the air conditioner has been turned on?

$t(10) = 1.5(\cos 15(10)) + 20$
$= 21.05$

c. What is the period of the function? Interpret this value in this context?

$\text{Period} = \dfrac{360}{b} = \dfrac{360}{15} = 24 \rightarrow 24$ minutes

ASSIGN: p. 528, #1 – 15, 17 – 19

It takes 24 minutes to complete 1 cycle. Max → max or min → min