M3201 - Section 6.1 and 6.2

UNIT 6 Exponential Functions

6.1/6.2: Characteristics of Exponential Functions

We will explore exponential functions of the form: $y = a(b)^x$

the variable is in the exponent

where $b > 1$ or $0 < b < 1$ and $a > 0$

Investigation:

1. a. Complete the table of values and sketch the graph of: $y = 2^x$

$y = 2^x$

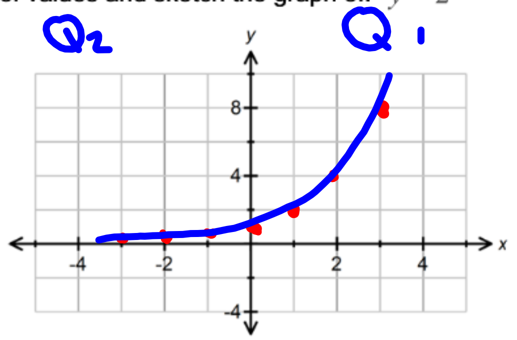
$y = 2^{-3}$

$= \left(\frac{1}{2}\right)^3$

$= \frac{1}{8}$

$= 0.125$

x	y
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8



$y = 2^{-2}$

$= \left(\frac{1}{2}\right)^2$

$= \frac{1}{4}$

$= 0.25$

b. Identify the characteristics.

x-intercept	none
y-intercept	1
End Behaviour	Q2 to Q1
Increasing/Decreasing	increasing
Horizontal Asymptote	$y = 0$
Domain	$x \in \mathbb{R}$
Range	$\{y \mid y > 0, y \in \mathbb{R}\}$

c. Compare the pattern in the table of values with the b-value.

$b > 1 \rightarrow$ increasing

→

1

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> All exponential functions of the form $y = a(b)^x$ gets very close to the x-axis but will never touch or cross it.
>
> This line that the graph gets close to is called the **horizontal asymptote** and has an equation of y = 0.

$\left(\frac{1}{2}\right)^{-3}$

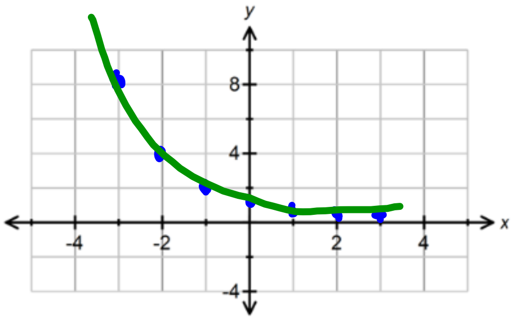
$= 2^3 = 8$

$\left(\frac{1}{2}\right)^{-2}$

$= 2^2 = 4$

2. a. Complete the table of values and sketch the graph of: $y = \left(\frac{1}{2}\right)^x$

x	y
-3	8
-2	4
-1	2
0	1
1	0.5
2	0.25
3	0.125



b. Identify the characteristics.

{ }

x-intercept	none
y-intercept	1
End Behaviour	Q2 to Q1
Increasing/Decreasing	Decreasing
Horizontal Asymptote	y = 0
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \mid y > 0, y \in \mathbb{R}\}$

c. Compare the pattern in the table of values with the b-value.

$0 < b < 1 \rightarrow$ decreasing

3. How are the graphs of $y = (2)^x$ and $y = \left(\frac{1}{2}\right)^x$ alike and how are they different?

Alike

Same x-int
end behaviour
horizontal asymptope
domain/range
y-int

Different

increasing/decreasing
b-value

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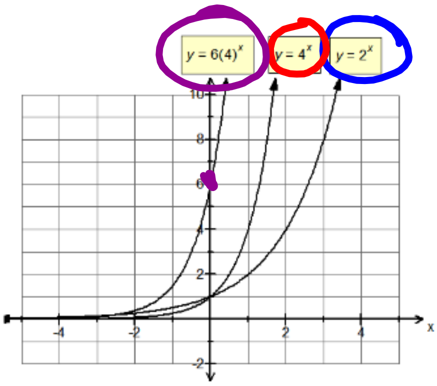
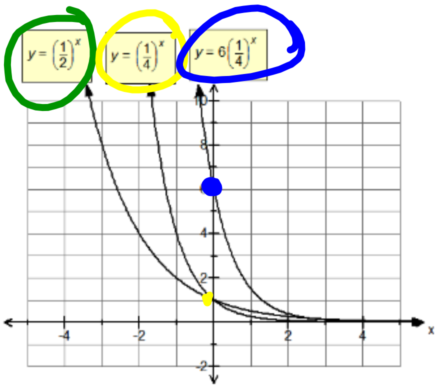
$y = 1(2)^x$

4. Given the following graphs in the form: $y = a(b)^x$



Graph A



Graph B

a. Complete the table.

	a	b	y-intercept	increasing/ decreasing
$y = 2^x$	1	2	1	inc
$y = 4^x$	1	4	1	inc
$y = 6(4)^x$	6	4	6	inc
$y = \left(\frac{1}{2}\right)^x$	1	1/2	1	dec
$y = \left(\frac{1}{4}\right)^x$	1	1/4	1	dec
$y = 6\left(\frac{1}{4}\right)^x$	6	1/4	6	dec

$b > 1$

$0 < b < 1$

b. Compare the **a-value** with the **y-intercept**.

What conclusion can you make?

a-value = y-int

c. Compare the **b-value** with the **shape** of the graph.

What conclusion can you make?

If $b > 1 \rightarrow$ increasing

If $0 < b < 1 \rightarrow$ decreasing

→

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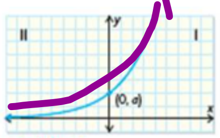
Characteristics of Exponential Functions of the Form:

$y = a(b)^x$ where $a > 0$ and $0 < b < 1$ or $b > 1$

- the number of x-intercepts: none
- y-intercept = a
- end behaviour: extends from Q2 to Q1
- equation of asymptote: $y = 0$
- domain: $x \in R$
- range: $y > 0$
- as x-values increase by 1, the y-values will increase/ decrease by a constant ratio equal to the b-value
- if b > 1 OR if 0 < b < 1

increasing from Q~~1~~ 2 to Q1



$a > 0, b > 1$

decreasing from Q2 to Q1

$a > 0, 0 < b < 1$

$y = a(b)^x$

1. What will happen if $b = 1$?

x	0	1	2	3	4
y	1	1	1	1	1

$y = (1)^0$ $(1)^1$ $(1)^2$

2. What will happen if $b = 0$?

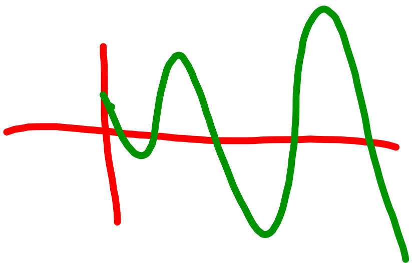
x	0	1	2	3	4
y	0	0	0	0	0

$y = 0^0 = 0$ $0^2 = 0$

$0^1 = 0$

3. What will happen if $b < 0$?

x	0	1	2	3	4
y	1	-2	4	-8	16



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$b = -2$

$y = (-2)^0 = 1$ $y = (-2)^1 = -2$ $y = (-2)^2 = 4$ $y = (-2)^3 = -8$ $y = (-2)^4 = 16$

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Journal Question:

How are the functions $y = x^2$ and $y = 2^x$ alike/different?

$y = x^2$

x	y
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
4	$4^2 = 16$

$y = 2^x$

x	y
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$

→ increasing
↳ diff rate

→ exponential
diff y-int

Example 1:

State the characteristics of each exponential function.

a) $y = 9\left(\frac{2}{3}\right)^x$ $0 < \frac{2}{3} < 1$

x-intercept	none
y-intercept	9
end behaviour	Q2 to Q1
inc/dec	dec
Equation of Asymptote	$y=0$
Domain	$x \in \mathbb{R}$
Range	$y>0, y \in \mathbb{R}$

b) $y = \frac{1}{2}(3)^x$ $3 > 1$

x-intercept	none
y-intercept	1/2
end behaviour	Q2 to Q1
inc/dec	inc
Equation of Asymptote	$y=0$
Domain	$x \in \mathbb{R}$
Range	$y>0, y \in \mathbb{R}$

c) $y = e^x$ $= 1(e)^x$

x-intercept	none
y-intercept	1
end behaviour	Q2 to Q1
inc/dec	inc
Equation of Asymptote	$y=0$
Domain	$x \in \mathbb{R}$
Range	$y>0, y \in \mathbb{R}$

NOTE:
e is a constant known as Euler's number.
e = 2.718....

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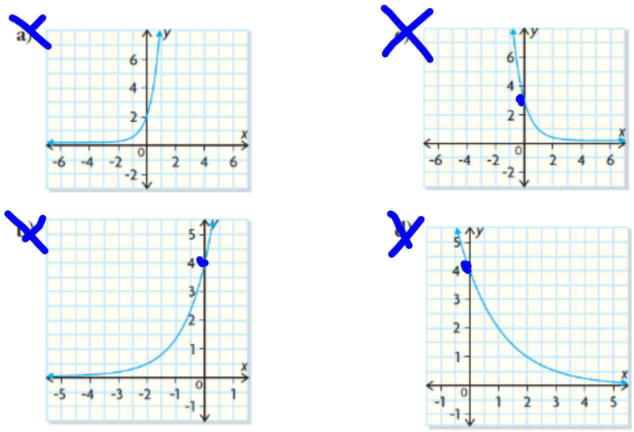
Example 2: (Ex. 3, p. 343)

Which exponential function matches each graph below? Explain why.

i) $y = 3(0.2)^x$ dec C ii) $y = 4(3)^x$ inc B iii) $y = 4(0.5)^x$ dec D iv) $y = 2(4)^x$ inc A

inc dec



inc dec

YOUR TURN: p. 345

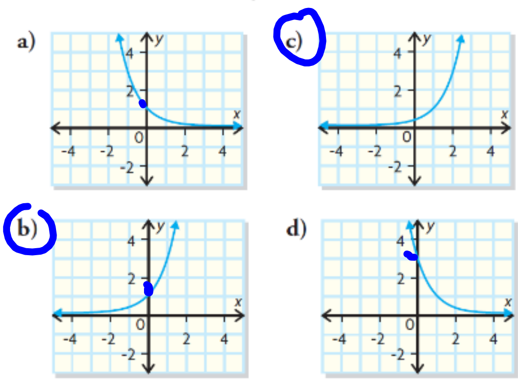
Which exponential function matches each graph below? Explain why.

i) $y = 1(3)^x$ B

ii) $y = \frac{1}{3}(3)^x$ C

iii) $y = 3\left(\frac{1}{3}\right)^x$ D

iv) $y = 1\left(\frac{1}{3}\right)^x$ A



Practice:

p. 347 - 350, #3, 4abcd, 5abcd, 9, 11, 12ace, 13, 15

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Section 6.3: Solving Exponential Equations

Exponent Laws	***Examples:***		
1. Zero Exponent: $b^0 = 1$	a) $\left(\frac{2}{3}\right)^0$ $= 1$	b) $5x^0$ $= 5(1)$ $= 5$	c) $(5x^2y^3)^0$ $= 1$
2. Negative Exponent: $b^{-x} = \frac{1}{b^x}$	a) 3^{-2} $= \frac{1}{3^2}$ $= \frac{1}{9}$	b) $\left(\frac{3}{4}\right)^{-2}$ $= \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$ $= \frac{16}{9}$	c) $\frac{1}{x^{-2}}$ $= \frac{x^2}{1}$ $= x^2$
3. Product Rule: $b^x \cdot b^y = b^{x+y}$ Multiply	a) $2^2 \times 2^3$ $= 2^{2+3}$ $= 2^5$	b) $x^4 \bullet x^2$ $= x^{4+2}$ $= x^6$	c) $5x^2y^4(3x^3y^2)$ $= 15x^{2+3}y^{4+2}$ $= 15x^5y^6$

exponent

2^2

base

$x^2 \cdot y^3$
$= x^2y^3$

$x^2 \cdot x^3$
$= x^{2+3}$
$= x^5$

1

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$x^{-2} = \frac{1}{x^2}$

Exponent Laws	Examples:		
Divide **4. Quotient Rule:** $\frac{b^x}{b^y} = b^{x-y}$	a) $\frac{x^5}{x^3}$ $= x^{5-3}$ $= x^2$	b) $\frac{12x^2}{4x^{-3}}$ $= 3x^{2-(-3)}$ $= 3x^{2+3}$ $= 3x^5$	c) $\frac{16x^3y^7}{8x^5y^4}$ $= 2x^{3-5}y^{7-4}$ $= 2x^{-2}y^3$ $= 2\frac{1}{x^2}y^3$ $= \frac{2y^3}{x^2}$
Exponents 5. Power Rule: $(b^x)^y = b^{xy}$ $(2x^2y^3)^2$ $2^2x^4y^6$ $4x^4y^6$	a) $(2^5)^2$ $= 2^{5\cdot2}$ $= 2^{10}$	b) $(2x^{-2})^3$ $= 2^3x^{-2\cdot3}$ $= 8x^{-6}$ $= 8 \cdot \frac{1}{x^6}$ $= \frac{8}{x^6}$	c) $\left(\frac{1}{3x^5}\right)^{-2}$ $= (3x^5)^2$ $= 3^2x^{5\cdot2}$ $= 9x^{10}$
6. Rational Exponents: $b^{\frac{1}{n}} = \sqrt[n]{b}$ $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$ $b^{\frac{1}{n}} = \sqrt[n]{b}$ $9^{\frac{1}{2}} = \sqrt[2]{9}$ $= 3$	a) $9^{\frac{1}{2}}$ $= \sqrt[2]{9}$ $= 3$	b) $64^{\frac{2}{3}}$ $= \left(\sqrt[3]{64}\right)^2$ $= (4)^2$ $= 16$	c) $\left[\left(\frac{16}{9}\right)^{-3}\right]^{\frac{1}{2}}$ $-3\cdot\frac{1}{2}$ $= \left(\frac{16}{9}\right)^{-3\cdot\frac{1}{2}}$ $= \left(\frac{16}{9}\right)^{-3/2}$ $= \left(\frac{9}{16}\right)^{3/2}$ $= \left(\sqrt[2]{\frac{9}{16}}\right)^3$ $= \left(\frac{3}{4}\right)^3$ $= \frac{3^3}{4^3} = \boxed{\frac{27}{64}}$

$b^{m/n} = \left(\sqrt[n]{b}\right)^m$

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7. Common Base Rule:

$b^x = b^y$ if and only if $x = y$

a) $2^x = 2^3$

$x = 3$

b) $5^{x+3} = 5^4$

$x + 3 = 4$

$x = 4 - 3$

$\boxed{x = 1}$

Example 1:

Express each of the following as a power with a base of 2.

a) 8

$= 2^3$

b) $\frac{1}{16}$

$= \frac{1}{2^4}$

$= 2^{-4}$

c) 8^{-2}

$= (2^3)^{-2}$

$= 2^{3\cdot(-2)}$

$= 2^{-6}$

$= \frac{1}{2^6}$

d) $8^{\frac{2}{3}}\left(\sqrt{16}\right)^3$

$= 2^{3\left(\frac{2}{3}\right)}(4)^3$

$\frac{\cancel{3}}{1}\cdot\frac{2}{\cancel{3}} = 2$

$= 2^2(2^2)^3$

$= 2^2\, 2^{2\cdot3}$

$= 2^2\, 2^6$

$= 2^{2+6} = 2^8$

Your Turn

Express each of the following as a power with a base of 3.

a) 27^2

$= (3^3)^2$

$= 3^{3\cdot2}$

$= 3^6$

b) $\left(\frac{1}{9}\right)^4$

$= \left(\frac{1}{3^2}\right)^4$

$= (3^{-2})^4$

$= 3^{-8}$

c) $27^{\frac{2}{3}}\left(\sqrt[3]{81}\right)^6$

$= 3^{3\left(\frac{2}{3}\right)}\left(\sqrt[3]{3^4}\right)^6$

$= 3^2\, 3^{\frac{4}{3}\cdot 6}$

$= 3^2\, 3^8$

$= 3^{10}$

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To Solve Exponential Equations:
- write both sides of the equation with the same base
- equate the exponents

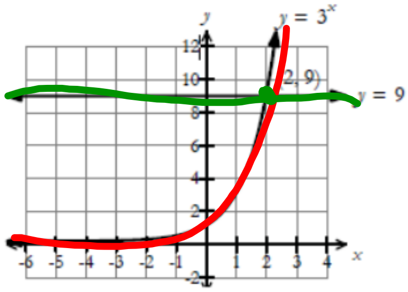
Example 2:

Solve for x: $3^x = 9$

$3^x = 3^2$

$x = 2$

The solution for the equation $3^x = 9$ can also be depicted graphically. We treat each side of the equation as 2 separate functions. The x-value of the point of intersection is the solution to the equation, $x = 2$.

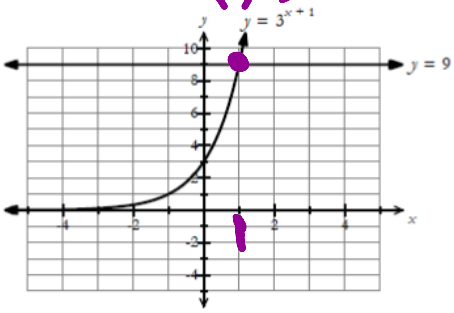


Example 3:

a) Use the graph to determine the solution for $3^{x+1} = 9$.

$x = 1$

(1,9)



b) Verify the solution algebraically.

$3^{x+1} = 9$

$3^{x+1} = 3^2$

$x+1 = 2$

$x = 2-1$

$\boxed{x = 1}$

→

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Example 4:

Solve each equation algebraically:

Rewrite each equation with the same base and equate the exponents.

a) $2^{x-1} = 16$

$2^{x-1} = 2^{4}$

$x - 1 = 4$

$x = 4 + 1$

$\boxed{x = 5}$

b) $4^{2x} = 8^{2x-3}$

$2^{2(2x)} = 2^{3(2x-3)}$

$2(2x) = 3(2x-3)$

$4x = 6x - 9$

$-2x = -9$

$\boxed{x = \frac{9}{2}}$

c) $4(3^{x+2}) = 36$

$\frac{4(3^{x+2})}{4} = \frac{36}{4}$

$3^{x+2} = 9$

$3^{x+2} = 3^{2}$

$x + 2 = 2$

$\boxed{x = 0}$

d) $8^{3x-4} + 7 = 71$

$8^{3x-4} = 71 - 7$

$8^{3x-4} = 64$

$8^{3x-4} = 8^{2}$

$3x - 4 = 2$

$\frac{3x}{3} = \frac{6}{3}$

$\boxed{x = 2}$

Your Turn:

Algebraically determine the solution for each of the following equations:

a) $3^{2x+1} = 3^{x+2}$

$2x + 1 = x + 2$

$2x - x = 2 - 1$

$\boxed{x = 1}$

b) $4^{3x+5} = 2^{4x-3}$

$2^{2(3x+5)} = 2^{4x-3}$

$2(3x+5) = 4x - 3$

$6x + 10 = 4x - 3$

$6x - 4x = -3 - 10$

$\frac{2x}{2} = \frac{-13}{2}$

$x = -\frac{13}{2}$

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c) $\cancel{3}(2)^{3x-2} = 48$

$\frac{\cancel{3}(2)^{3x-2}}{3} = \frac{48}{3}$

$2^{3x-2} = 16$

$2^{3x-2} = 2^{4}$

$3x-2=4$

$\frac{3x}{3} = \frac{6}{3}$

$\boxed{x=2}$

d) $9(2^{3x+5}) - 8 = 28$

$9(2^{3x+5}) = 28+8$

$\frac{\cancel{9}(2^{3x+5})}{9} = \frac{36}{9}$

$2^{3x+5} = 4$

$2^{3x+5} = 2^{2}$

$3x+5=2$

$3x = 2-5$

$\frac{\cancel{3}x}{3} = \frac{-3}{3}$

$\boxed{x=-1}$

Example 5:

Solve each equation algebraically:

Fraction in the base = Negative Exponent

a) $5^x = \frac{1}{125}$

$5^x = \frac{1}{5^3}$

$5^x = 5^{-3}$

$\boxed{x=-3}$

b) $(32)^{x-2} = \left(\frac{1}{4}\right)^{5x-3}$

$32^{x-2} = 4^{-1(5x-3)}$

$2^{5(x-2)} = 2^{2(-1)(5x-3)}$

$5(x-2) = -2(5x-3)$

$5x-10 = -10x+6$

$15x = 16$

$x = \frac{16}{15}$

Your Turn:

a) $\left(\frac{1}{8}\right)^{x-3} = 16^{2x+1}$

$\left(\frac{1}{2^3}\right)^{x-3} = 2^{4(2x+1)}$

$2^{-3(x-3)} = 2^{4(2x+1)}$

$-3(x-3) = 4(2x+1)$

$-3x+9 = 8x+4$

$\frac{-11x}{-11} = \frac{-5}{-11}$

$x = \frac{5}{11}$

b) $2(4)^{2x} = \frac{1}{32}$

$2^1(2^2)^{2x} = \frac{1}{2^5}$

$2^{3(2x)} = 2^{-5}$

$3(2x) = -5$

$\frac{\cancel{6}x}{\cancel{6}} = \frac{-5}{6}$

$\boxed{x = -\frac{5}{6}}$

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$$\sqrt{x} = \sqrt[2]{x} = x^{1/2}$$
$$\sqrt[3]{x} = x^{1/3}$$

Example 6:

Radical = Fractional Exponent

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Solve each equation algebraically:

a) $\sqrt{8} = 2^{3x-4}$

$$8^{1/2} = 2^{3x-4}$$
$$2^{3(1/2)} = 2^{3x-4}$$
$$3(1/2) = 3x-4$$
$$\frac{3}{\cancel{2}}\cdot\cancel{2} = (3x-4)\cdot 2$$
$$3 = 6x-8$$
$$11 = 6x$$
$$\boxed{x = \frac{11}{6}}$$

b) $5^{x+2} = \sqrt[3]{25}$

$$5^{x+2} = \sqrt[3]{5^2}$$
$$5^{x+2} = 5^{\frac{2}{3}}$$
$$3\cdot(x+2) = \frac{2}{\cancel{3}}\cdot\cancel{3}$$
$$3x+6 = 2$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{-4}{3}$$
$$\boxed{x = \frac{-4}{3}}$$

Your Turn:

a) $27^{2x-1} = \sqrt[3]{3^{2x}}$

$$3^{3(2x-1)} = 3^{\frac{2x}{3}}$$
$$3(2x-1)\cdot 3 = \frac{2x}{\cancel{3}}\cdot\cancel{3}$$
$$9(2x-1) = 2x$$
$$18x-9 = 2x$$
$$18x-2x = 9$$
$$\frac{\cancel{16}x}{\cancel{16}} = \frac{9}{16}$$
$$\boxed{x = \frac{9}{16}}$$

b) $\sqrt[2]{3^x} = 9^{2x+1}$

$$3^{\frac{x}{2}} = 3^{2(2x+1)}$$
$$\frac{x}{\cancel{2}}\cdot\cancel{2} = 2(2x+1)\cdot 2$$
$$x = 4(2x+1)$$
$$x = 8x+4$$
$$x-8x = 4$$
$$\frac{\cancel{-7}x}{\cancel{-7}} = \frac{4}{-7}$$
$$\boxed{x = -\frac{4}{7}}$$

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Example 7:

Solve each equation algebraically:

a) $9^{x-1} \times 81^{2x-1} = 27^{3x-2}$

$3^{2(x-1)} \times 3^{4(2x-1)} = 3^{3(3x-2)}$

$2(x-1) + 4(2x-1) = 3(3x-2)$

$2x - 2 + 8x - 4 = 9x - 6$

$2x + 8x - 9x = -6 + 2 + 4$

$\boxed{x = 0}$

b) $\dfrac{64^{x-1}}{16^{2x+2}} = 2^{x-2}$

$\dfrac{2^{6(x-1)}}{2^{4(2x+2)}} = 2^{x-2}$

$6(x-1) - 4(2x+2) = x - 2$

$6x - 6 - 8x - 8 = x - 2$

$6x - 8x - x = -2 + 6 + 8$

$-3x = 12$

$\dfrac{-3x}{-3} = \dfrac{12}{-3}$

$\boxed{x = -4}$

c) $5^{x^2+2x} = 125$

$5^{x^2+2x} = 5^3$

$x^2 + 2x = 3$

$x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$\boxed{x = 1}$ $\boxed{x = -3}$

$-1 + 3 = 2$

$-1 \cdot 3 = -3$

→

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Your Turn:

a) $4^{3x+2} \times 32^{x-2} = 8^{3x-4}$

$2^{2(3x+2)} \times 2^{5(x-2)} = 2^{3(3x-4)}$

$2(3x+2) + 5(x-2) = 3(3x-4)$

$6x + 4 + 5x - 10 = 9x - 12$

$6x + 5x - 9x = -12 - 4 + 10$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{-6}{2}$

$\boxed{x = -3}$

b) $\frac{125^{2x+1}}{625^{x+2}} = 3125^{x+2}$

$\frac{5^{5(2x+1)}}{5^{6(x+2)}} = 5^{7(x+2)}$

$5(2x+1) - 6(x+2) = 7(x+2)$

$10x + 5 - 6x - 12 = 7x + 14$

$10x - 6x - 7x = 14 + 5 - 12$

$\frac{-\cancel{3}x}{-\cancel{3}} = \frac{31}{-3}$

$\boxed{x = \frac{-31}{3}}$

Example 8: Identify and correct the error.

$\sqrt{5} = 25^{3x+4}$

$5^{\frac{1}{2}} = 5^{2(3x+4)}$

$5^{\frac{1}{2}} = 5^{6x+4}$ error

$\frac{1}{2} = 6x + 4$

$2 = 12x + 8$

$-6 = 12x$

$-\frac{1}{2} = x$

$5^{1/2} = 5^{6x+8}$

$2 \cdot \left(\frac{1}{2}\right) = (6x+8) \cdot 2$

$1 = 12x + 16$

$1 - 16 = 12x$

$\frac{-15}{12} = \frac{\cancel{12}x}{\cancel{12}}$

$x = -\frac{15}{12}$

$x = -\frac{5}{4}$

Practice:

p. 361, #2abcd, 4cdef, 5abc, 7bdf

9

M3201 - Section 6.3

Recall that an exponential expression arises when a quantity changes by the same factor for each unit of time.

For example,

- a population doubles every year;
- a bank account increases by 0.1% each month;
- a mass of radioactive substance decreases by 1/2 every 462 years.

The ***half-life*** exponential function is given by the equation:

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where $A(t)$ is the value after time, t

A_0 is the initial value

h is the **half-life**

The ***doubling*** exponential function is given by the equation:

$$A(t) = A_0\left(2\right)^{\frac{t}{d}}$$

where $A(t)$ is the value after time, t

A_0 is the initial value

d is the **doubling time**

→

10

M3201 - Section 6.3

Example 9:

h

The half-life of a radioactive isotope is 30 hours. The amount of radioactive isotope A(t), at time t, can be modelled by the function:

A(t) $\quad A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ $\quad A_0$

Algebraically determine how long it will take for a sample of 1792 mg to decay to 56 mg.

Solution:

$$\frac{56}{1792 \div 56} = \frac{\cancel{1792}\left(\frac{1}{2}\right)^{\frac{t}{30}}}{1792}$$

$$\frac{1}{32} = \left(\frac{1}{2}\right)^{t/30}$$

$$\frac{1}{2^5} = \left(\frac{1}{2}\right)^{\frac{t}{30}}$$

$$\left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{t/30}$$

$$(5)30 = \left(\frac{t}{30}\right)30$$

$$\boxed{150 = t}$$

Example 10:

The population of trout growing in a lake can be modeled by the function $P(t) = 200(2)^{\frac{t}{5}}$ where $P(t)$ represents the number of trout and t represents the time in years after the initial count. How long will it take for there to be 6400 trout?

Note:
- the value of 200 represents the initial number of trout
- the number of trout doubles every 5 years

Solution:

$$\frac{6400}{200} = \frac{\cancel{200}(2)^{t/5}}{200}$$

$$32 = 2^{(t/5)}$$

$$2^{5} = 2^{\frac{t}{5}}$$

$$5 \times 5 = \frac{t}{\cancel{5}} \times \cancel{5}$$

$$25 = t$$

$$\boxed{t = 25}$$

It will take 25 years to get 6400 trout.

11

M3201 - Section 6.3

Your Turn:

The half life of Radon 222 is 92 hours. From an initial sample of 48g, how long would it take to decay to 6g?

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$\frac{6}{48} = \frac{\cancel{48}}{\cancel{48}}\left(\frac{1}{2}\right)^{\frac{t}{92}}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{92}}$$

$$\frac{1}{2^3} = \left(\frac{1}{2}\right)^{\frac{t}{92}}$$

$$\left(\frac{1}{2}\right)^{3} = \left(\frac{1}{2}\right)^{\frac{t}{92}}$$

$$3 \cdot 92 = \frac{t}{\cancel{92}} \cdot \cancel{92}$$

$$276 = t$$

$$\boxed{t = 276}$$

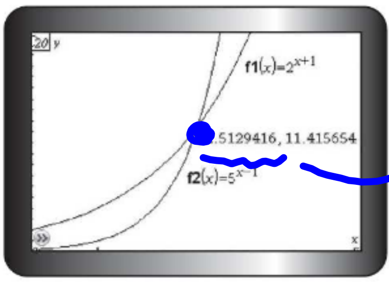
It would take 276 hours.

Example 11: (ex. 4, p. 359)

Solve: $2^{x+1} = 5^{x-1}$

Since neither base can be written as a power of the other, we can't equate the exponents. We will learn how to solve this algebraically in Unit 7. For now, we will solve by using graphing technology.

$y_1 = 2^{(x+1)}$ $\quad y_2 = 5^{(x-1)}$



Solution: x ~ 2.5

Practice:

p. 363 - 365, #11a, 15, 16b

+ worksheet

12

M3201 - Section 6.4

Section 6.4: Modelling Data Using Exponential Functions

In the previous unit, linear, quadratic and cubic regressions were performed on polynomial functions. Similarly, technology can be used to create a scatter plot and determine the equation of the exponential regression function that models the data.

Example 1: (ex. 1, p. 371)

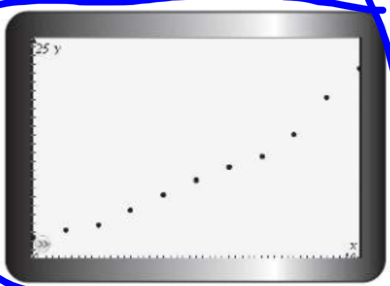
The population of Canada from 1871 to 1971 is shown in the table below. In the third column, the values have been rounded.

Year	Actual Population of Canada	Population of Canada (millions)
1871	2 436 297	2.44
1881	3 229 633	3.23
1891	3 737 257	3.74
1901	5 418 663	5.42
1911	7 221 662	7.22
1921	8 800 249	8.80
1931	10 376 379	10.38
1941	11 506 655	11.51
1951	14 009 429	14.01
1961	18 238 247	18.24
1971	21 568 305	21.57

Statistics Canada

a) Using graphing technology, create a graphical model and an algebraic exponential model for the data.

Let x represent the number of 10-year intervals since 1871. Let y represent the population of Canada in millions.

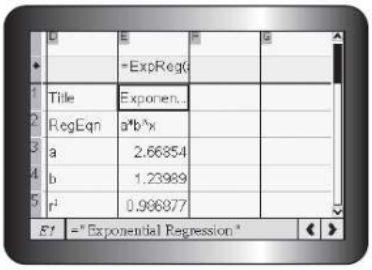


As the x-values get larger, the y-values also grow larger, but not at a constant rate.

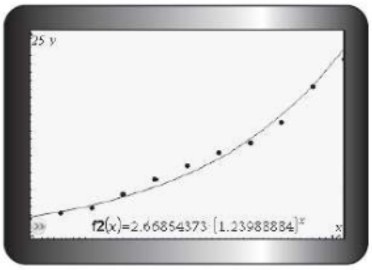
The data can be modelled by an ***exponential growth function.***

1

M3201 - Section 6.4



An exponential regression on the data determines the equation of the curve of best fit.



The equation is verified by graphing it on the same grid as the given points.

The exponential equation that models the data is:

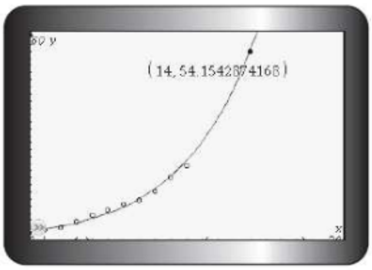
$$y = a(b)^x$$
$$y = 2.67(1.24)^x$$

b) Assuming that the population growth continued at the same rate to 2011, estimate the population in 2011. Round your answer to the nearest million.

Since x represents the number of 10-year intervals since 1871,

$$x = \frac{2011 - 1871}{10} = 14$$

$$y = 2.67(1.24)^x$$
$$y = 2.67(1.24)^{14}$$
$$y = 54.25$$
$$y = 54 \text{ million}$$



The solution can also be *extrapolated* from the graph.

M3201 - Section 6.4

Example 2: (p. 373)

Sonja did an experiment to determine the cooling curve of water. She placed the same volume of hot water in three identical cups. Then she recorded the temperature of the water in each cup as it cooled over time. Her data for three trials is given as follows.

Trial 1

Time (min)	Temperature (°C)
0	80
5	69
10	61
20	45
30	34
40	26

Trial 2

Time (min)	Temperature (°C)
0	75
5	66
10	59
20	44
30	32
40	23

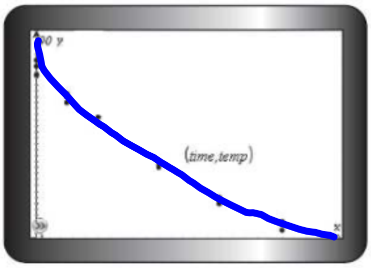
Trial 3

Time (min)	Temperature (°C)
0	78
5	68
10	61
20	44
30	33
40	25

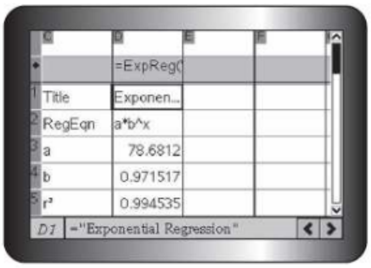
a) Construct a scatter plot to display the data. Determine the equation of the exponential regression function that models Sonja's data.

Let x represent the time, in minutes, since the experiment began.

Let y represent the temperature in degrees Celsius.



The data can be modelled by an ***exponential decay function.***

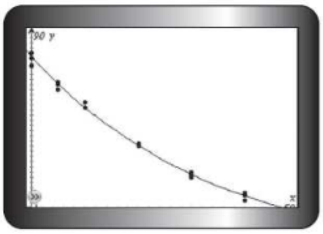


An exponential regression on the data determines the equation of the curve of best fit.

Note: The initial temperatures of the three samples were not the same: 80 °C, 75 °C, and 78 °C.
The regression model defines a = 78.68, which is close to all three initial values.

3

M3201 - Section 6.4



The equation is verified by graphing it on the same grid as the given points.

The exponential equation that models the data is:

$$y = a(b)^x$$

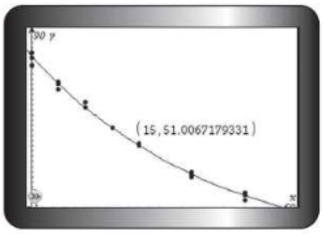
$$y = 78.68(0.97)^x$$

The solution can also be *extrapolated* from the graph.

b) Estimate the temperature of the water 15 min after the experiment began. Round your answer to the nearest degree.

After 15 minutes, $y = 78.68(0.97)^{15}$

$$\boxed{y = 50^0 C}$$

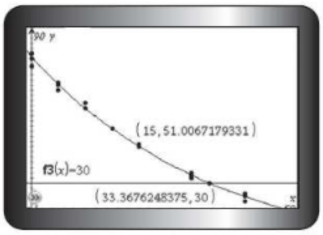


The solution can also be *interpolated* from the graph.

c) Estimate when the water reached a temperature of 30 °C. Round your answer to the nearest minute.

$$y = 78.68(0.97)^x$$

$$30 = 78.68(0.97)^x$$



The point of intersection is (33.367..., 30),

so x = 33 minutes.

Practice Questions:

p. 377-382, #4c, 5c, 8c
(use the graphs and regression equations from p. 740)

M3201 - Section 6.5

Section 6.5: Financial Applications Involving Exponential Functions

1. Simple Interest:

Simple interest is calculated only in terms of the original amount invested, not on the accumulated interest.

Simple Interest Formula: $A = P + Prt$ or $A = P(1 + rt)$

where P is the principal amount

t is the time in years

r is the interest rate per annum (as a decimal)

$4\% = 0.04$

NOTE: A is the sum of the principal (P) and the accumulated interest (Prt)

Example 1:

Kyle invested his summer earnings of $5000.00 at 8% simple interest, paid annually.

a) Create a table of values and graph the growth of the investment for 6 years using time, in years, as the domain and the value of the investment as the range.

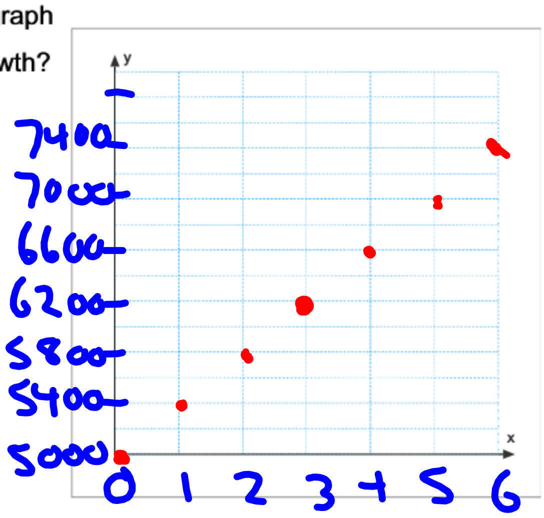
$A = 5000 + 5000(0.08)t$

Time (years)	Value of Investment ($)
0	5000
1	$A = 5000 + 5000(0.08)(1) = 5400$
2	$A = 5000 + 5000(0.08)(2) = 5800$
3	. . . . $= 6200$
4	6600
5	7000
6	7400

1

M3201 - Section 6.5

a) What does the shape of the graph tell you about the type of growth?



Linear

→ growing at a constant rate

b) Why is the data discrete?

Only get paid at the end of every year.

c) What do the y-intercept and slope represent for the investment?

y-int: Principal amount

slope: 400 → interest earned

d) What is the value of the investment after 10 years?

$A = P + Prt$

$= 5000 + 5000(0.08)10$

$= 5000 + 4000$

$= 9000$

e) How much interest was earned after 10 years?

Total − Principal

9000 − 5000

= $4000 of interest

→

2

M3201 - Section 6.5

2. Compound Interest:

Compound interest is determined by applying the interest rate to the sum of the principal and any accumulated interest.

Compound Interest Formula: $\boxed{A = P(1+i)^n}$

where *A* is the future value

P is the principal amount

i is the interest rate **per compounding period** (expressed as a decimal)

t is the time in years

n is the **number of compounding periods**

n is NOT the number of years!

Refer to previous example of \$5000.00 (*P*) in a savings account earning an annual interest of 8%.

Time (years)	Amount of Annual Interest	Value of Investment (\$)
0	0	$A=5000(1+0.08)^0 = 5000$
1	400	$A=5000(1+0.08)^1 = 5400$
2	432	$A=5000(1+0.08)^2 = 5832$
3	467	$A=5000(1+0.08)^3 = 6299$

NOTE: The accumulated interest and the value of the investment do not grow by a constant amount as they do with simple interest.

An exponential regression to model the investment would result in the equation: $y = 5000(1.08)^x$

Note how this compares to: $A = P(1+i)^n$.

They are the same!

3

M3201 - Section 6.5

Investments can also have daily, weekly, monthly, quarterly, semi-annually, or annually compounding periods.

KNOW

Compounding Period	Number of Times Interest is Paid	Interest Rate per Compounding Period, i
Daily	365 times per year	$i = \frac{\text{annual rate}}{365}$
Weekly	52 times per year	$i = \frac{\text{annual rate}}{52}$
Bi-Weekly	26 times per year	$i = \frac{\text{annual rate}}{26}$
Semi-monthly	24 times per year	$i = \frac{\text{annual rate}}{24}$
Monthly	12 times per year	$i = \frac{\text{annual rate}}{12}$
Quarterly	4 times per year	$i = \frac{\text{annual rate}}{4}$
Semi-annually	2 times per year	$i = \frac{\text{annual rate}}{2}$
Annually	1 time per year	$i = \frac{\text{annual rate}}{1}$

KNOW

bi-annually → once every 2 years

For example, if \$5000 is invested at 6% compounded monthly,

$$i = \frac{\text{annual rate}}{12} = \frac{0.06}{12} = \boxed{0.005}$$

The compound interest formula is defined as:

$$A = 5000(1.005)^n$$

where n is the number of months, NOT the number of years

4

M3201 - Section 6.5

$= 0.048$

Example 2: Complete the table if the interest rate is 4.8% per year.

Compounding Period	Number of Times Interest is Paid	Interest Rate per Compounding Period (i)
Bi-Monthly	6	$i = \frac{0.048}{6} = 0.008$
Monthly	12	$i = \frac{0.048}{12} = 0.004$
Quarterly	4	$i = \frac{0.048}{4} = 0.012$
Semi-Annually	2	$i = \frac{0.048}{2} = 0.24$
Annually	1	$i = \frac{0.048}{1} = 0.048$

Example 3:

If $5000 is invested, calculate A (the future value) using $A = P(1+i)^n$ for each situation.

a) 11% per year, compounded quarterly for 3 years

$= 0.11$ 4

$A = P(1+i)^n$

$A = 5000(1.0275)^{12}$

$= \boxed{6924}$

Best

$i = \frac{0.11}{4} = 0.0275$

$n = 4 \times 3$

$\boxed{n = 12}$

b) 6.5% per year, compounded semi-annually for 3 years

$= 0.065$ 2

$A = 5000(1.0325)^6$

$= \boxed{6058}$

$i = \frac{0.065}{2} = 0.0325$

$n = 2 \times 3$

$\boxed{n = 6}$

c) 15.6% per year, compounded monthly for 2 years

12

$A = 5000(1.013)^{24}$

$= \boxed{6817}$

$i = \frac{0.156}{12}$

$= 0.013$

$n = 12 \times 2$

$\boxed{n = 24}$

→

5

M3201 - Section 6.5

Example 4:

$3000 was invested at 6% per year compounded monthly.

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a) Write the exponential function in the form: $A = P(1+i)^n$

$A = 3000(1+0.005)^n$ $\quad i = \frac{0.06}{12} = 0.005$

b) What will be the future value of the investment after 4 years?

$A = 3000(1.005)^{48}$

$= \boxed{3811}$

$n = 4 \times 12$

$\boxed{n = 48}$

Example 5:

An automobile that originally costs $24 000 loses one-fifth of its value each year. What is the value after 6 years?

losing

$\frac{1}{5} = 0.2 \rightarrow \boxed{-0.2}$

$i = -0.2$

$n = 6$

$A = P(1+i)^n$

$= 24000(1 + -0.2)^6$

$= 24000(0.8)^6$

$= \boxed{6291}$

Example 6:

$2000 is invested for 3 years at an annual interest rate of 9% $= 0.09$ compounded monthly. Lucas solved the following equation:

$A = 2000(1.0075)^3$

error

Correct the error and solve the problem.

$i = \frac{0.09}{12} = 0.0075$

$A = P(1+i)^n$

$= 2000(1.0075)^{36}$

$= \boxed{2617}$

$n = 3 \times 12$

$\boxed{n = 36}$

6

M3201 - Section 6.5

Example 7:

Nora is about to invest $5000 in an account that pays 6% interest a year compounded monthly for the next 3 years. A different financial institution offers 6.5% interest a year compounded semi-annually for the next 3 years. Write a function that models the growth of Nora's investment for each situation. Should Nora invest her money in this financial institution instead? Explain why or why not.

$$A = 5000(1.005)^{36} = \boxed{5983}$$

$$i = \frac{0.06}{12} = 0.005$$

$$n = 3 \times 12 = 36$$

$$A = 5000(1.0325)^{6} = \boxed{6058}$$

$$i = \frac{0.065}{2} = 0.0325$$

$$n = 3 \times 2 = 6$$

Best! She should change to a different financial institution.

Practice Questions:

p. 396 – 397, #10, 14

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