Math 3201 ***UNIT 5: Polynomial Functions*** **NOTES**

Section 5.1 and 5.2:

Characteristics of Graphs and Equations of Polynomials Functions

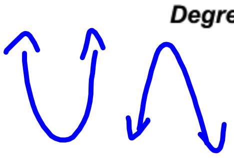
What is a polynomial function?

***Polynomial Function*:**

- A function that contains only the operations of multiplication and addition with real-number coefficients, whole-number exponents, and two variables.

Polynomial functions are named according to their ***degree***.

Degree of a Function:



- Is equal to the ***highest*** exponent in the polynomial.

Examples of Polynomials:

Type of Polynomial	Standard Form	Specific Example	Degree	Graph
Constant	$f(x) = a$	$f(x) = 5x^0 = 5$	0	Horizontal line
Linear	$f(x) = ax + b$	$f(x) = 2x^1 + 1$	1	Line with slope *a*
Quadratic	$f(x) = ax^2 + bx + c$	$f(x) = 2x^2 - x + 1$	2	Parabola
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	$f(x) = 2x^3 + 3x^2 - 2x + 1$	3	?? Will explore!

NOTES:

1. The terms in a polynomial function are normally written so that the powers are in ***descending*** order.
2. Vertical lines are not considered here because they are not functions since they do not satisfy the vertical line test. Vertical lines are not part of the polynomial family!

In previous grades, we have already explored Constant (Gr.9), Linear (Gr. 9 and L1), and Quadratic (L2) functions. We will review these functions and then explore cubic functions.

First however, we will need a few new terms:

Constant Term:

- The term in the polynomial that does not have a variable.

Leading Coefficient:

- The coefficient (the number in front) of the term with the greatest degree in a polynomial function in standard form. This is denoted as ***a***.
- For example, the leading coefficient of $f(x) = 2x^3 + 3x^2 - 4x + 1$ is 2.

Question 1:

For each polynomial, determine the degree, leading coefficient and constant term.

a) $f(x) = 3x^2 - 4x + 5$

Degree = 2 Leading coefficient = 3 Constant term = 5

b) $f(x) = -2x + 7$

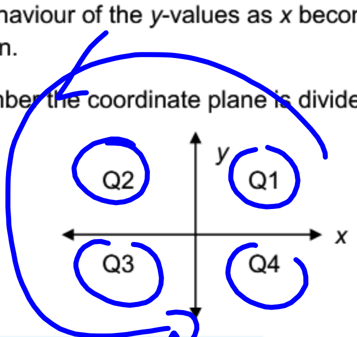
Degree = 1 Leading coefficient = -2 Constant term = 7

c) $f(x) = x^3 + 4x^2 - 6x - 9$

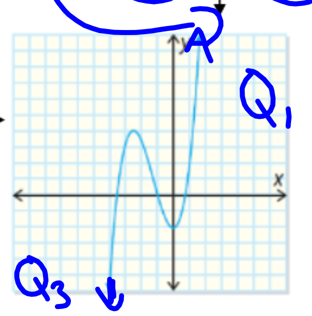
Degree = 3 Leading coefficient = 1 Constant term = -9

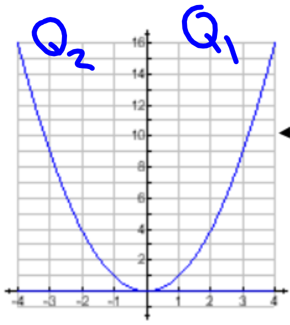
End Behaviour:

- The description of the shape of the graph, ***from left to right***, on the coordinate plane.
- The behaviour of the *y*-values as *x* becomes large in the positive or negative direction.
- Remember the coordinate plane is divided into four ***quadrants***.



This cubic function extends from Q3 to Q1



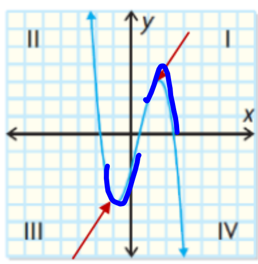


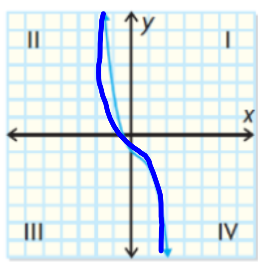
This quadratic function extends from Q2 to Q1

Turning Point:

- Any point where the graph of a function changes from increasing to decreasing or from decreasing to increasing.
- For example,

This curve has 2 turning points



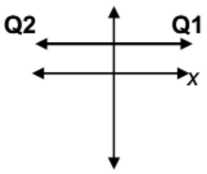
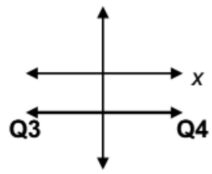


This curve does not have any turning points

So let's review!

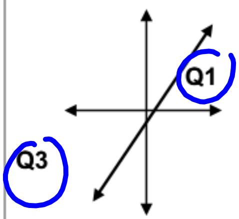
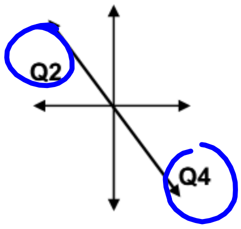
1. ***Constant Function,*** $f(x) = a$ ***(Degree = 0)***

$f(x)=4$
$y=4$

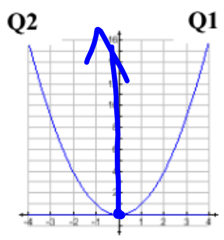
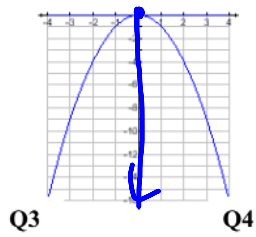
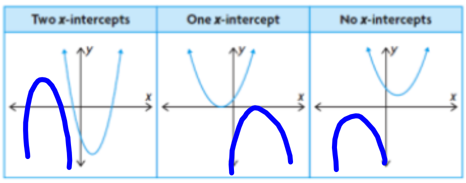
Possible Sketches **Horizontal Lines**	If a is positive ($a > 0$)  	If a is negative ($a < 0$) 
End Behaviour	Line extends from Q2 to Q1	Line extends from Q3 to Q4
Number of *x*-intercepts	0 (exception: $y = 0$ in which case every point is on the x-axis)	
Number of *y*-intercepts	1 ($y = a$)	
Number of Turning Pts	0	
Domain	$\{x \mid x \in \Re\}$	
Range	$\{y \mid \underline{y = a}, y \in \Re\}$	

$\{y \mid y=4, y \in R\}$

2. ***Linear Functions,*** $f(x) = ax + b$ ***(Degree = 1)***

Possible Sketches **Oblique Lines**	If leading coefficient, *a*, is positive ($a > 0$) 	If leading coefficient, *a*, is negative ($a < 0$) 
End Behaviour	Line falls to the left and rises to the right Line extends from Q3 to Q1	Line falls to the right and rises to the left Line extends from Q2 to Q4
Number of *x*-intercepts	1	
Number of *y*-intercepts	1 ($y = b$)	
Number of Turning Pts	0	
Domain	$\{x \mid x \in \Re\}$	
Range	$\{y \mid y \in \Re\}$	

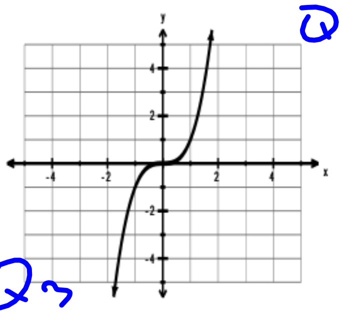
3. ***Quadratic Functions,*** $f(x) = ax^2 + bx + c$ ***(Degree = 2)***

Possible Sketches **Parabolas**	If leading coefficient, *a*, is positive ($a > 0$) Parabola opens up and has a minimum  	If leading coefficient, *a*, is negative ($a < 0$) Parabola opens down and has a maximum 
End Behaviour	Parabola rises to the left and rises to the right Extends from Q2 to Q1	Parabola falls to the left and falls to the right Extends from Q3 to Q4
	NOTE: The graphs have the SAME behavior to the left and right.	
Range (depends on vertex and opening)	$\{y \mid y \geq \min, y \in \Re\}$	$\{y \mid y \leq \max, y \in \Re\}$
Number of *x*-intercepts	 2, 1 or 0	
Number of *y*-ints	1 ($y = c$)	
Number of Turning Points	1	
Domain	$\{x \mid x \in \Re\}$	

Let's now explore Cubic Functions of the form $y = ax^3 + bx^2 + cx + d$!

1. $y = x^3$

Q1



Q3

Leading Coefficient: 1

Constant Term: 0

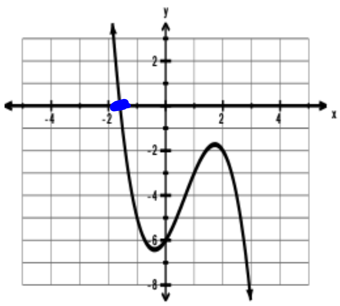
End Behaviour: Q3 to Q1

# of Turning Points: 0

# of x-intercepts: 1

y-intercept: y = 0 (0, 0)

2. $y = -x^3 + 2x^2 + 2x - 6$



Leading Coefficient: −1

Constant Term: −6

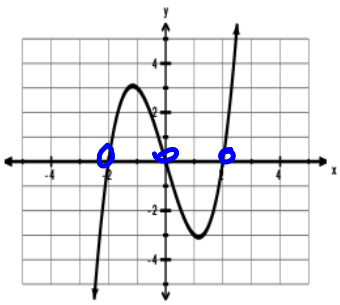
End Behaviour: Q2 to Q4

# of Turning Points: 2

# of x-intercepts: 1

y-intercept: y = −6 (0, −6)

3. $y = x^3 - 4x$



Leading Coefficient: 1

Constant Term: 0

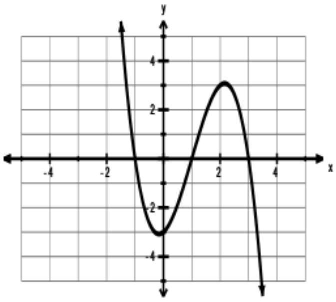
End Behaviour: Q3 to Q1

# of Turning Points: 2

# of x-intercepts: 3

y-intercept: y = 0 (0, 0)

4. $y=-x^3+3x^2+x-3$



Leading Coefficient: -1

Constant Term: -3

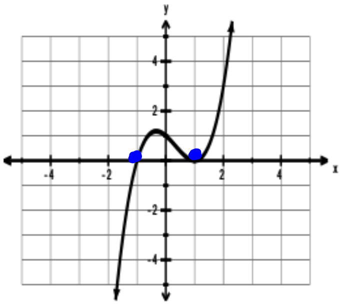
End Behaviour: Q2 to Q4

# of Turning Points: 2

# of x-intercepts: 3

y-intercept: y = -3 (0, -3)

5. $y=x^3-x^2-x+1$



Leading Coefficient: 1

Constant Term: 1

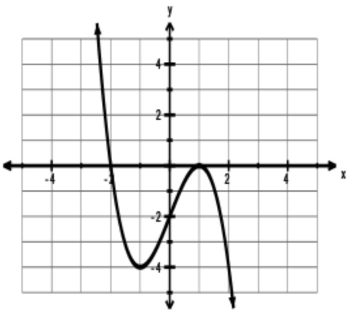
End Behaviour: Q3 to Q1

# of Turning Points: 2

# of x-intercepts: 2

y-intercept: y = 1 (0, 1)

6. $y=-x^3+3x-2$



Leading Coefficient: -1

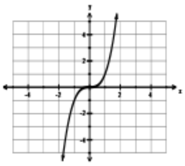
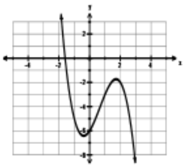
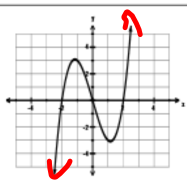
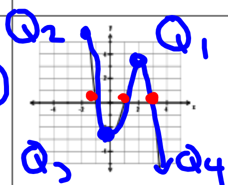
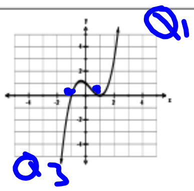
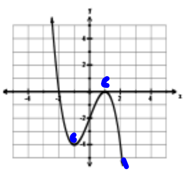
Constant Term: -2

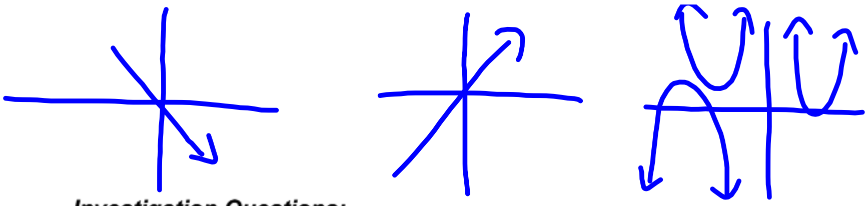
End Behaviour: Q2 to Q4

# of Turning Points: 2

# of x-intercepts: 2

y-intercept: y = -2 (0, -2)

Function	Sketch	Leading Coeff.(*a*)	Cons. Term (d)	End Behav.	Number Turn. Pts	# of x-ints.	y-int.
a) $y = x^3$		1	0	Q3 to Q1	0	1	(0,0)
b) $y = -x^3 + 2x^2 + 2x - 6$		-1	-6	Q2 to Q4	2	1	(0,-6)
c) $y = x^3 - 4x$		1	0	Q3 to Q1	2	3	(0,0)
d) $y = -x^3 + 3x^2 + x - 3$		-1	-3	Q2 to Q4	2	3	(0,-3)
e) $y = x^3 - x^2 - x + 1$		1	1	Q3 to Q1	2	2	(0,1)
f) $y = -x^3 + 3x - 2$		-1	-2	Q2 to Q4	2	2	(0,-2)



Investigation Questions:

(Cubic)

1. How are the sign of the leading coefficient and the end behavior related?

   (−) Negative (+) positive

   Q_2 to Q_4 Q_3 to Q_1

2. How are the degree of the function and the number of *x*-intercepts related?

   Deg
   0 → 0 x-int
   1 → 1 x-int
   2 → 0, 1, 2 x-int

   Deg 3 → 1, 2, 3 x-int

   Degree = max # of x-int

3. How is the *y*-intercept related to the equation?

   constant term = y-int

   True for all functions

4. How many turning points can a cubic function have?

   0 or 2

5. In general, how are the number of turning points and the degree related?

   max # of turning points is 1 less than our degree

6. What is the domain and range for cubic functions?

   Range $\{y \mid y \in \mathbb{R}\}$
   Domain $\{x \mid x \in \mathbb{R}\}$

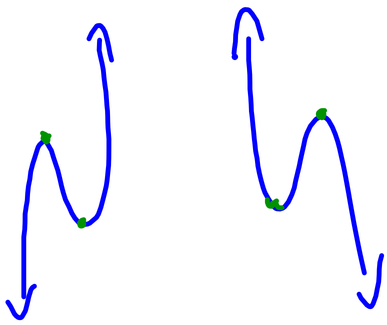
   Same for linear

7. Explain why quadratic functions have maximum or minimum values, but cubic polynomial functions have only turning points?

Quadratic:

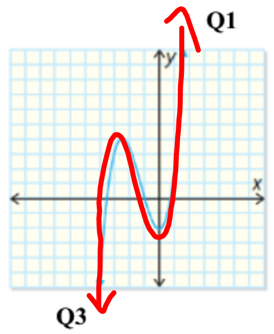
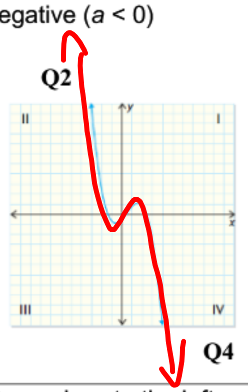
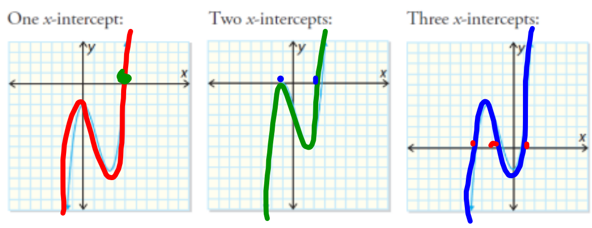
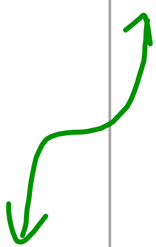
Quadratics have a vertex that is a true max/min

Cubic:

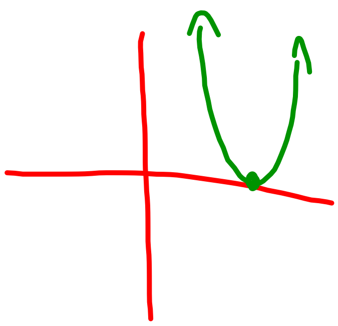


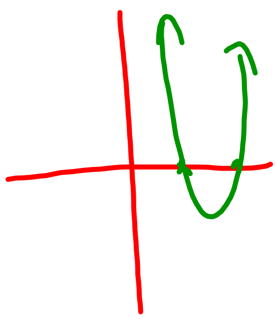
Cubics have a local max/min

4. *Cubic Functions,* $f(x) = ax^3 + bx^2 + cx + d$ ***(Degree = 3)***

Possible Sketches **Sideways "S"**	If leading coefficient, *a,* is positive ($a > 0$) 	If leading coefficient, *a,* is negative ($a < 0$) 
End Behaviour	Curve falls to the left and rises to the right Extends from Q3 to Q1	Curve rises to the left and falls to the right Extends from Q2 to Q4
	NOTE: The graphs have OPPOSITE behaviors to the left and right. They have similar behavior to the behavior of linear functions.	
Number of *x*-intercepts		

Number of *y*-intercepts	1 ($y = d$)	**Domain**	$\{x \mid x \in \Re\}$
Number of Turning Points	2 or 0	**Range**	$\{y \mid y \in \Re\}$

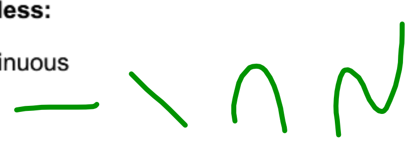




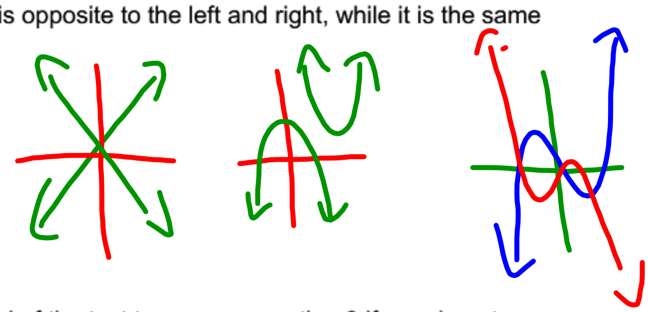
***** Refer to page 276 and page 286 for a summary.*****

Summary Points for Polynomials of Degree 3 or less:

- The graph of a polynomial function is continuous
- Degree determines the shape of graph



- Degree = max # of *x*-intercepts
- There is only one *y*-intercept for every polynomial and it is equal to the constant term
- The maximum number of turning points is one less than the degree. That is, a polynomial of degree *n*, will have a maximum of *n* – 1 turning points.
- The end behavior of a line or curve is the behavior of the *y*-values as *x* becomes large in the positive or negative direction. For linear and cubic functions the end behavior is opposite to the left and right, while it is the same for quadratic functions.



ASSIGN: **p. 277, #1 – 4**

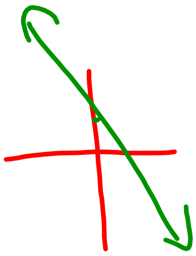
(You can refer to the graphs at the end of the text to answer question 3 if you do not have graphing technology)

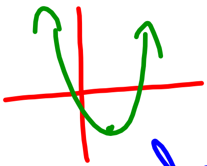
$f(0.75) = 4(0.75)^2 - 6(0.75) + 3$
$= -5.25$

EXAMPLES:

$x = \frac{-b}{2a} = \frac{-(-6)}{2(4)} = \frac{6}{8} = 0.75$

1. Determine the following characteristics of each function:
   a) number of possible x-intercepts
   b) y-intercept
   c) domain and range
   d) number of possible turning point
   e) end behavior





a. $f(x) = -3x + 2$ Linear

Degree: 1

a) 1
b) y-int = 2
c) $\{x \mid x \in \mathbb{R}\}$
$\{y \mid y \in \mathbb{R}\}$
d) 0
e) Q_2 to Q_4

b. $f(x) = 4x^2 - 6x + 3$

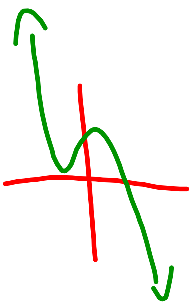
degree: 2 quadratic

a) 0, 1, 2
b) (0, 3) or y-int = 3
c) $\{x \mid x \in \mathbb{R}\}$
$\{y \mid y \geq \min, y \in \mathbb{R}\}$
$\{y \mid y \geq -5.25, y \in \mathbb{R}\}$
d) 1
e) Q_2 to Q_1

c. $f(x) = -2x^3 - 3x^2 + 2x - 1$

degree: 3 cubic

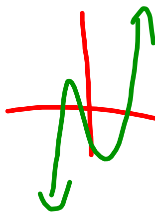
a) 1, 2, 3
b) (0, -1)
c) $\{x \mid x \in \mathbb{R}\}$ ← domain
$\{y \mid y \in \mathbb{R}\}$ ← range
d) 0, 2
e) Q_2 to Q_4



d. $f(x) = 4x^3 + 2x$

degree: 3 → cubic

a) 1, 2, 3
b) y-int = 0 or (0, 0)
c) $\{x \mid x \in \mathbb{R}\}$
$\{y \mid y \in \mathbb{R}\}$
d) 0, 2
e) Q_3 to Q_1



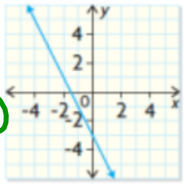
2. Match each graph with the correct polynomial function.

cubic $a(x) = -x^3 + 4x^2 - 2x - 2$ $\qquad$ $b(x) = x^2 - 2x - 2$ quad

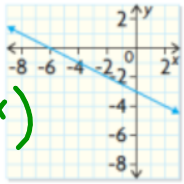
cubic $c(x) = x^3 - 2x^2 - x - 2$ $\qquad$ $d(x) = -\frac{1}{2}x - 3$ linear

quad $e(x) = x^2 - 2x + 1$ $\qquad$ $f(x) = -2x - 3$ linear

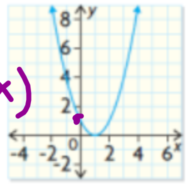
i) f(x)



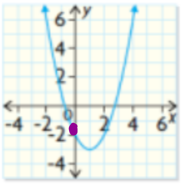
iii) d(x)



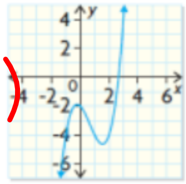
v) e(x)



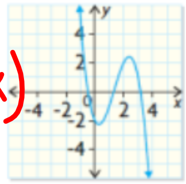
ii) b(x)



iv) c(x)

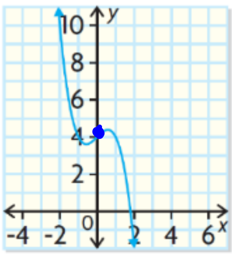


vi) a(x)



3. Determine the degree, the sign of the leading coefficient, and the constant term for the polynomial function represented by each graph.

cubic

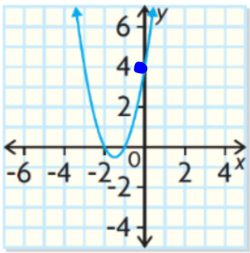


a.

Degree: 3

Sign: negative

Constant Term: 4



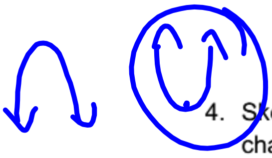
quad

b.

Degree: 2

Sign: positive

Constant Term: 4

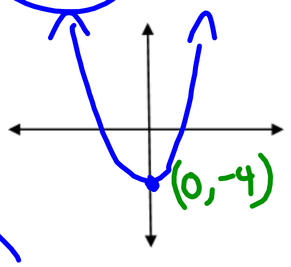


4. Sketch a possible graph of polynomial functions that satisfy each set of characteristics.

quad

a. Degree 2, one turning point which is a minimum, constant term of – 4

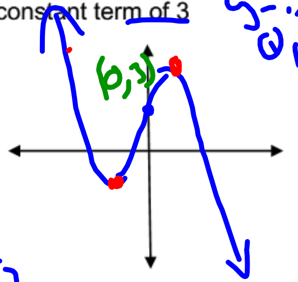
y-int



cubic

b. Two turning points (one in Q3 and one in Q1), negative leading coefficient, constant term of 3

y-int
Q1

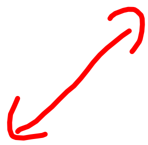


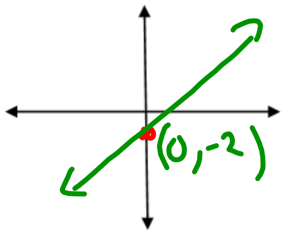
Q3

linear

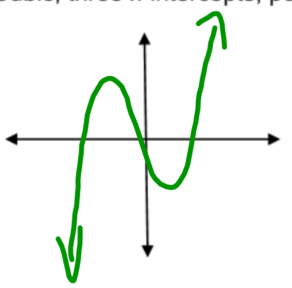
c. Degree 1, positive leading coefficient, constant term of – 2

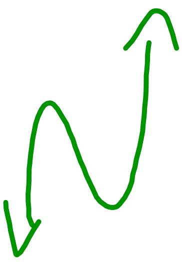
y-int





d. Cubic, three *x*-intercepts, positive leading coefficient

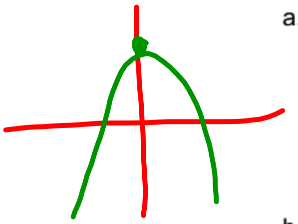




5. Write a polynomial function that satisfies each set of characteristics.

Q3 Q4

a. Extending from QIII to QIV, one turning point, y-intercept of 5



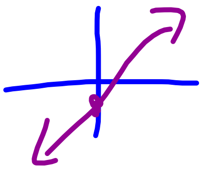
$y = -x^2 + x + 5$

$y = -5x^2 + 3x + 5$

b. Extending from QIII to QI, y-intercept of -4

Q3 Q1

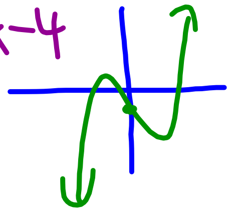
Linear



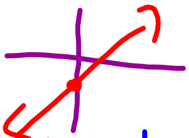
$y = 2x - 4$

Cubic

$y = x^3 + 2x^2 + x - 4$



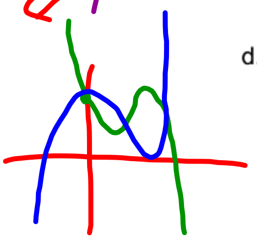
c. Degree 1, increasing, y-intercept of -3



$y = 2x - 3$

$y = x - 3$

d. Two turning points, y-intercept of 7

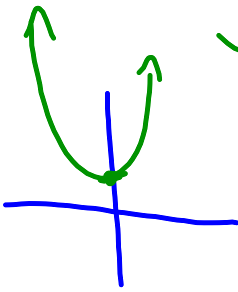


$y = x^3 + x^2 + x + 7$

$y = x^3 + 7$

e. Range of $y \geq 2$ and y-intercept of 2

quadratic



$y = x^2 + x + 2$

or

$y = 5x^2 - 2x + 2$

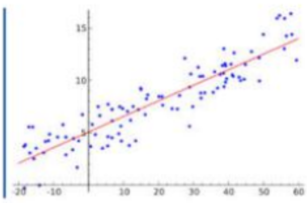
ASSIGN:

p. 287, #1 – 4, 6 – 13

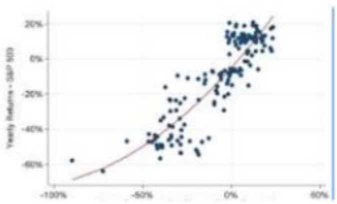
Section 5.3 and 5.4:

Modelling Data with a Line/Curve of Best Fit

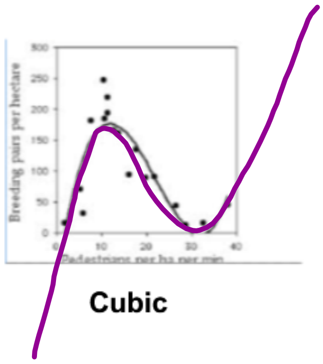
A **line/curve of best fit** is a straight line/curve that best approximates the trend in a **scatter plot**.



Linear



Quadratic



Cubic

Regression results in an equation that balances the points in the scatter plot on both sides of the line/curve.

A line/curve of best fit can be used to predict values that are not recorded or plotted.

Predictions can be made, either within the data (**interpolation**) or outside the data (**extrapolation**), by reading values from the line/curve of best fit on a scatter plot or by using the equation of the line/curve of best fit.

When using a regression model, however, caution should be used in the range of the data since predicting values too far outside the data may be inappropriate and yield unrealistic answers.

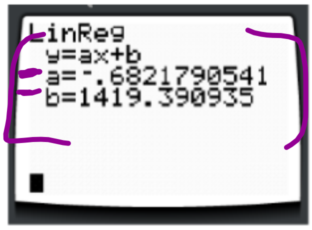
Example 1:

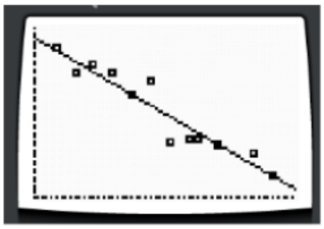
The winning times for the men's 20 km biathlon in the Winter Olympics from 1964 to 2010 (except for 2002) are shown in the table.

Year	1964	1968	1972	1976	1980	1984	1988	1992	1994	1998	2006	2010
Winning Time (min)	80.4	73.8	75.9	74.2	68.3	71.9	56.6	57.6	57.4	56.2	54.3	48.4

One graphing technology that can be used to help determine an equation of the line or curve of best fit is a graphing calculator.

Here are a few screen shots from a graphing calculator to show the data.





linear

a) Write the equation of the line of best fit.

$y = mx + b$

$y = ax + b$

$y = -0.6822x + 1419.39$

b) Using the equation of the line of best fit, determine a possible winning time for the event in the 2002 Winter Olympics to the nearest hundredth of a minute.

↳ sub 2002 in for x

$y = -0.6822(2002) + 1419.39$

$= 53.63$ minutes

c) Compare your estimate with the actual winning time of 51.0 min in 2002.

Close but not exact

Good predictor

Example 2:

Consider the data in the table.

6.7 12.53

x	0	2	4.5	5.2	9.5	12
y	5.1	6.7	8.2	8.8	11.9	13.4

Using technology, the equation of the line of best fit was determined to be:

$$\boxed{y = 0.693x + 5.183}$$

a) Determine, to the nearest tenth, the value of *y* when *x* is 10.6.

$y = 0.693(10.6) + 5.183$ ↳ sub 10.6 for x

$= \boxed{12.53}$

b) Determine, to the nearest tenth, the value of *x* when *y* is 9.8

$9.8 = 0.693(x) + 5.183$ sub $y = 9.8$

$9.8 - 5.183 = 0.693x$

$\dfrac{4.617}{0.693} = \dfrac{\cancel{0.693}x}{\cancel{0.693}}$

$\boxed{x = 6.66}$

Example 3:

Matt buys T-shirts for a company that prints art on T-shirts and then resells them. When buying the T-shirts, the price Matt must pay is related to the size of the order. Five of Matt's past orders are listed in the table below.

Number of Shirts	Cost per Shirt ($)
500	3.25
700	1.95
200	5.20
460	3.51
740	1.69

Matt has misplaced the information from his supplier about price discounts on bulk orders. He would like to get the price per shirt below $1.50 on his next order.

Creating a scatter plot and completing a linear regression, using technology, the equation for the linear regression function to model the data was determined to be:

price per shirt → $P = -0.0065n + 6.5$

a) What do the slope and y-intercept of the equation of the linear regression function represent in this context?

Slope = −0.0065 → tells us that the price will decrease

y-int = 6.5 → starting cost of shirts is $6.50

b) Use the linear regression function to extrapolate the size of order necessary to achieve the price of $1.50 per shirt.

Sub P = 1.50

$1.5 = -0.0065n + 6.5$

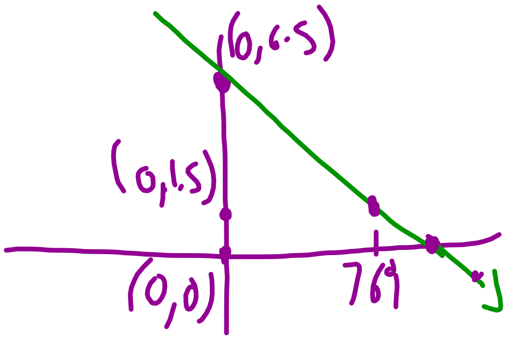
$1.5 - 6.5 = -0.0065n$

$\frac{-5}{-0.0065} = \frac{-0.0065n}{-0.0065}$

$\boxed{n = 769}$

ASSIGN: p. 301, #3, 4, 6, 9, 10

Needs to order ~~about~~/at least 769 shirts



Example 4:

speed

Audrey is interested in how speed plays a role in car accidents. She knows that there is a relationship between the speed of a car and the distance needed to stop. She has found some experimental data (p.308) on a reputable website, and she would like to write a summary for the graduation class website.

Creating a scatter plot of this data she noticed a quadratic pattern and therefore completed a quadratic regression on the data which yielded the equation:

$$\boxed{y = 0.008x^2 + 0.539x - 10.449}$$

a) Use the equation to compare the stopping distance at 30 km/h with the stopping distance at 50 km/h, to the nearest tenth of a meter.

30 km/h

$y = 0.008(30)^2 + 0.539(30) - 10.449$

$= 0.008(900) + 16.7 - 10.449$

$= \boxed{12.92 \text{ m}}$

50 km/h

$y = 0.008(50)^2 + 0.539(50) - 10.449$

$= \boxed{36.5 \text{ m}}$

b) Determine the maximum speed that a car should be travelling in order to stop within 4 m, the average length of a car.

(Note: Use the "trace" feature on a graphing calculator or read directly from a graph. Refer to p. 310 in the textbook for the graph.)

<u>Using trace:</u> to stop within 4 m the maximum speed is 20.5 km/h

or

sub $y = 4$

$4 = 0.008x^2 + 0.539x - 10.449$

$0 = 0.008x^2 + 0.539x - 14.449$

use quad. formula

$x = 20.386$

$\simeq 20.4$

Example 5:

The following table shows the average retail price of gasoline, per litre, for a selection of years in a 30-year period beginning in 1979.

Years after 1979	Price of Gas (¢/L)	Years after 1979	Price of Gas (¢/L)
0	21.98	17	58.52
1	26.18	20	59.43
2	35.63	22	70.56
3	43.26	23	70.00
4	45.92	24	74.48
7	45.78	25	82.32
8	47.95	26	92.82
9	47.53	27	97.86
12	57.05	28	102.27
14	54.18	29	115.29

Statistics Canada

a) Given the scatter plot, what polynomial function could be used to model the data? Explain.

Cubic → not a straight line
→ not a parabola

End behaviour: Q3 to Q1

b) The equation of the cubic regression function that models this data is $P = 0.0123n^3 - 0.4645n^2 + 6.295n + 23.452$ where P represents the average price of gas per litre and n represents the number of years after 1979. Use this equation to estimate the average price of gas in 1984 and 1985.

$$1984 - 1979 = 5$$

Sub $n = 5$

1984: $P = 0.0123(5)^3 - 0.4645(5)^2 + 6.295(5) + 23.452$

$= 1.5375 - 11.6 + 31.475 + 23.452$

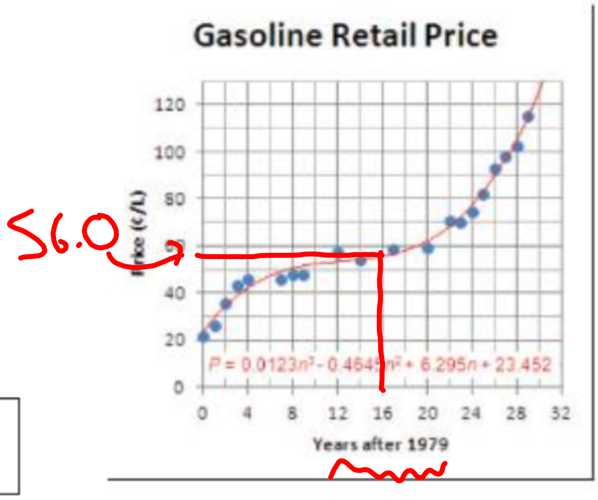
$= \boxed{44.8}$

1985: $\boxed{P = 47.2}$ ← Using our regression curve

c) Estimate the year in which the average price of gas was 56.0 cents/L.

$n = 16$

$1979 + 16$
$= 1995$



ASSIGN: p. 313, #2, 3abc, 4, 8abc

Extra Example: (if using graphing calculator)

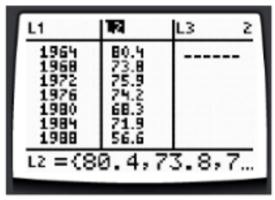
The concentration (in milligrams per litre) of a medication in a patient's blood is measured as time passes. Susan has collected the following data and is attempting to express the concentration as a polynomial function of time.

Time (T hours)	0	1.5	3	4.5	6	7.5	9
Concentration (C mg/L)	0	26.9	41.2	47.8	46.0	36.8	20.3

a) Use technology to create a scatter plot and determine a quadratic regression function that models the data. Round parameters *a*, *b*, and *c* to two decimal places.

b) The doctor has decided that the patient needs a second dose of medication when the concentration in the blood is less than 10 mg/L. If the first dose of medication was given at 9:00am, at what time should the second dose be given?

Key Strokes for Graphing Calculator for Example 1

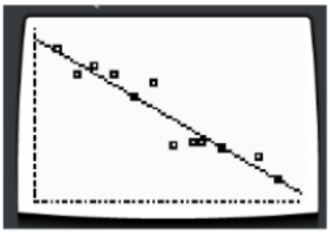
Step 1	• Hit STAT • Choose 1: Edit by either hitting 1 or ENTER If necessary clear out any old data in the list by highlighting L1 at the top of the list: press CLEAR ENTER (repeat process for L2)	 
Step 2	• Enter your data into L1 and L2	
Step 3	• Hit 2nd Y= [STAT PLOT] • Choose 1: Plot1 • Turn ON the plot by pressing ENTER	  

<table>
<tr>
<td>Step 4</td>
<td>• Press ZOOM 9(zoom stat)

This is the data value that you submitted into your lists in Step 1.</td>
<td></td>
</tr>
<tr>
<td>Step 5</td>
<td>• Hit STAT the arrow over to CALC
• Choose 4: LinReg(ax+b)
Once you have Lin Reg on your screen follow it with L1,L2,Y1

To get Y1: VARS arrow over to Y-VARS, choose 1: FUNCTION then choose Y1

Choosing Y1 will graph the line of best fit of the data you inputted in Step 1 onto your graph</td>
<td>


</td>
</tr>
<tr>
<td>Step 6</td>
<td>• Press GRAPH to see your line of best fit</td>
<td></td>
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