Math 3201 Notes
Chapter 3: Probability

Learning Goals: See p. 137 text.

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§3.1 **Exploring Probability (0.5 classes)**

Read *Goal* p. 140 text.

Outcomes:
1. Define **probability**. p. 141
2. Define **experimental probability**. p. 141
3. Define **theoretical probability**. p. 141
4. Explain the difference between **experimental probability** and **theoretical probability**. p. 141

Def$^{\underline{n}}$: Recall that in a previous math course, the **probability** (P) of an event (A) was defined as the number of favorable outcomes divided by the total number of possible outcomes. As a formula, we write

$$P(A) = \frac{\text{\# favorable outcomes}}{\text{\# possible outcomes}}$$

Also recall that the probability of an event can range from 0 (impossible) to 1 (certain) and that probability can be written as a fraction, a decimal, or a
percent.

Def$^{\underline{n}}$: The probability that an event does NOT occur $P(A')$ is given by the formula

$$P(A') = 1 - P(A)$$

E.g.: If the probability of rolling a 2 on a six-sided die is $\frac{1}{6}$ then

$$P(\text{not rolling } 2) = 1 - \frac{1}{6} = \frac{5}{6} = 0.8\bar{3} = 83.\bar{3}\%$$

Def$^{\underline{n}}$: The **experimental probability** of an event (A) is the number of times the event occurs during an experiment divided by the total number of trials.

E.g.: If Sam rolled a number cube 50 times and a 3 appeared 10 times, then the experimental probability of rolling a 3 is 10 out of 50 or 20%. This could also be written as

$$P(\text{rolling } 3) = \frac{10}{50} = \frac{1}{5} = 0.2 = 20\%$$

E.g.: A coin is tossed 60 times. 27 times head appeared. Find the experimental probability of getting heads.

a) $\frac{1}{27}$
b) $\frac{9}{20}$
c) $\frac{1}{60}$
d) $\frac{3}{20}$

Def^{n}: The **theoretical probability** of an event (A) is the number of times the event occurs during an experiment divided by the total number of trials when the number of trials becomes VERY large.

E.g.: If we toss a fair coin, what is the theoretical probability that a tail will show up?

$$P(\text{tossing a tail}) = \frac{1}{2} = 0.5 = 50\%$$

E.g.: A bag contains 20 marbles. 15 of them are red and 5 of them are blue in color. Find the theoretical probability of picking a red marble.

$$P(\text{red marble}) = \frac{15}{20} = \frac{3}{4} = 0.75 = 75\%$$

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§3.2 **Probability and Odds (2 classes)**

Read *Goal* p. 142 text.

Outcomes:
1. Define the **odds in favour** of an event. p. 142
2. Define the **odds against** an event. p. 142
3. Determine the odds in favour of an event. p. 142
4. Determine the odds against an event. p. 142
5. Given the probability of an event, determine the odds in favour of the event. p. 144
6. Given the probability of an event, determine the odds against the event. p. 144
7. Given the odds in favour of an event, determine the probability of the event. p. 144
8. Given the odds against an event, determine the probability of the event. p. 144

Def$^{\underline{n}}$: The **odds in favour** of an event is the number of favorable outcomes divided by the number of unfavorable outcomes.

$$\text{odds in favor} = \frac{\text{\# favorable outcomes}}{\text{\# unfavorable outcomes}}$$

E.g.: The odds in favour of rolling a 4 on a six-sided die is $\frac{1}{5}$ or $1:5$.

E.g.: The odds in favour of selecting an "E" from the word ICEBERG is $\frac{2}{5}$ or $2:5$

Def$^{\underline{n}}$: The **odds against** an event is the number of unfavorable outcomes divided by the number of favorable outcomes.

$$\text{odds against} = \frac{\text{\# unfavorable outcomes}}{\text{\# favorable outcomes}}$$

E.g.: The odds against rolling a 4 on a six-sided die is $\frac{5}{1}$ or $5:1$.

E.g.: The odds against selecting an "E" from the word ICEBERG is $\frac{5}{2}$ or $5:2$

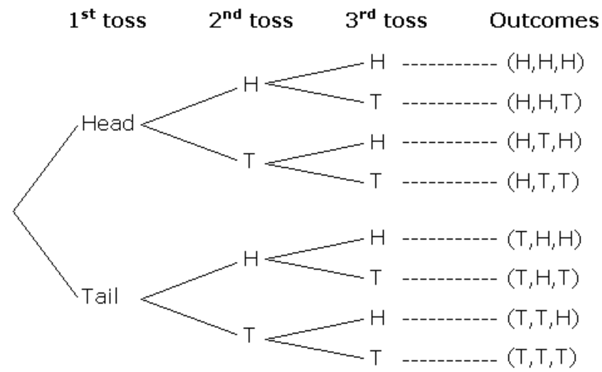
Note that the odds in favour of an event is the **reciprocal** of the odds against the event, and vice versa.

E.g.: The odds against Jeff passing in his next math assignment is 4:3. What are the odds in favour of Jeff passing in is next math assignment? ANS: 3:4

Sample Exam Question

Given the tree diagram showing the outcomes when a fair coin is tossed three times, determine the odds in favour of obtaining two heads and a tail.

a) $8:3$
b) $3:8$
c) $3:5$
d) $5:3$



Changing from Probability to Odds

E.g.: If the probability of having triplets is $\frac{1}{1000}$, what are the odds in favor of having triplets?

The number of favorable outcomes is 1 and the number of unfavorable outcomes is 999, so the odds in favor of having triplets is $\frac{1}{999}$ or $1:999$.

E.g.: If the probability of having triplets is $\frac{1}{1000}$, what are the odds against having triplets?

The number of favorable outcomes is 999 and the number of unfavorable outcomes is 1, so the odds in favor of having triplets is $\frac{999}{1}$ or $999:1$.

E.g.: The probability of drawing a red card from a standard deck is $\frac{26}{52}=\frac{1}{2}$. What are the odds against drawing a red card?

The number of unfavorable outcomes is 26 (26 black cards) and the number of unfavorable outcomes is 26 (26 red cards), so the odds in against drawing a red is $\frac{26}{26}$ or $1:1$.

E.g.: The probability of drawing 9 from a standard deck is $\frac{4}{52}$. What are the odds in favour of drawing a 9? What the odds against drawing a 9?

Odds in favour of drawing a 9 $=\frac{4}{48}=1:12$

Odds against drawing a $9 = \frac{48}{4} = 12:1$

E.g.: Complete the table below.

Probability of an Event	Odds in Favor of the Event
15%	$\frac{15}{85} = \frac{3}{17}$ or $3:17$
50%	
95%	

E.g.: Complete the table below.

Probability of an Event	Odds Against the Event
28%	$\frac{72}{28} = \frac{18}{7}$ or $18:7$
50%	
88%	

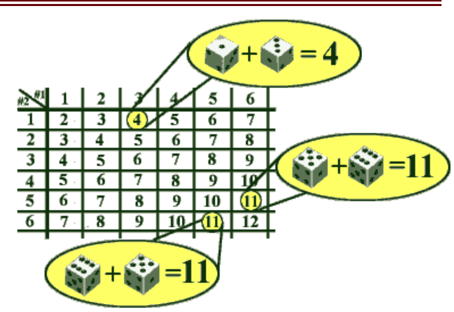
Changing from Odds to Probability

E.g.: The odds of rolling an even number on a six-sided die is 3:3. What is the probability of rolling an even number on a six-sided die?

Number of favorable outcomes = 3
Number of unfavorable outcomes = 3
Total number of outcomes = 6

$$P(\text{even number}) = \frac{3}{6} = \frac{1}{2} = 0.50 = 50\%$$

E.g.: The odds of a sum of 7 when two dice are rolled is 6:30. What is the probability a sum of 7 when two dice are rolled?



Number of favorable outcomes = 6
Number of unfavorable outcomes = 30
Total number of outcomes = 36

$$P(\text{sum of } 7) = \frac{6}{36} = \frac{1}{6} = 0.1\overline{6} = 16.\overline{6}\%$$

E.g.: The odds against of a sum of 10 when two dice are rolled is 33:3. What is the probability a sum of 10 when two dice are rolled?

Number of favorable outcomes = 3
Number of unfavorable outcomes = 33
Total number of outcomes = 36

$$P(\text{sum of } 10) = \frac{3}{36} = \frac{1}{12} = 0.08\bar{3} = 8.\bar{3}\%$$

E.g.: The odds against of a sum of 8 when two dice are rolled is 31:5. What is the probability of NOT getting a sum of 8 when two dice are rolled?

Number of favorable outcomes = 31
Number of unfavorable outcomes = 5
Total number of outcomes = 36

$$P(\text{not a sum of } 8) = \frac{31}{36} = 0.86\bar{1} = 86.\bar{1}\%$$

E.g.: Complete the table below.

Odds in Favor of an Event	Probability of the Event
3:5	$\frac{3}{8} = 0.375 = 37.5\%$
7:5	
100:1	

E.g.: Complete the table below.

Odds Against an Event	Probability of the Event
4:7	$\frac{7}{11} = 0.\overline{63} = 63.\overline{63}\%$
10:9	
2:1	

E.g: Two brown-eyed people are told that there is a 35% probability that their baby will have blue eyes. What are the odds in favour of such parents having a baby with blue eyes? What are the odds against?

$35\% = \frac{35}{100} = \frac{7}{20}$

Number of favourable outcomes is 7
Total number of outcomes is 20
Number of unfavorable outcomes is 20 – 7 = 13

Odds in favor of blue eyes are $\frac{7}{13}$ or 7:13

Odds against blue eyes are $\frac{13}{7}$ or 13:7

E.g.: Max has gone to the store to buy a pair of jeans. From experience, he knows the odds against the store having his style of jeans in his size are 10 : 32. Determine the probability that the store will have jeans in his size.

Number of unfavorable outcomes is 10.
Number of favourable outcomes is 32
Total number of outcomes is 42

$$\mathrm{P}\left(\text{correct size}\right) = \frac{32}{42} = 0.\overline{761904} = 76.\overline{190476}\%$$

Making a Decision Based on Odds and Probability

E.g.: An NHL hockey game must be decided in a shootout and the coach must decide on the order of the players in the shootout. The statistics for his three most successful shootout players are shown in the table below. What should be the order of the players?

Player	Attempts	Goals Scored
1	13	8
2	17	10
3	3	2

Odds in favour of Player 1 scoring are 8:5 or 1.6:1.
Odds in favour of Player 2 scoring are 10:7 or about 1.42:1
Odds in favour of Player 3 scoring are 2:1

The order of players should be 3, 1, 2.

OR

The probability that Player 1 scores is $\frac{8}{13} \approx 61.5\%$

The probability that Player 2 scores is $\frac{10}{17} \approx 58.8\%$

The probability that Player 3 scores is $\frac{2}{3} \approx 66.7\%$

The order of players should be 3, 1, 2.

E.g.: Laser printers are on sale at Staples. The last 9 times laser printers were on sale, they were in stock only 4 times.

a) Determine the odds in favour of laser printers being in stock this time. 4:5

b) Determine the odds against laser printers being in stock at this time. 5:4

E.g.: Ratings for the movie *Butterfly Dreaming* indicate that 45% of the viewers are male, 10% are under 18, 20% are 19-29 years old, 20% are 30 to 45 years old, and 50% are older than 45. Suppose that someone watching *Butterfly Dreaming*.

a) What are the odds in favor of this person being female? 55:45 or 11:9

b) What are the odds against this person being 45 or younger? 50:50 or 1:1

Read "Key Ideas", p. 147 text.
Read "Need to Know", p. 147 text.
Do #'s 1-3, 5, 7, 10, 11, 13, CYU/Practising pp.148-49 text in your homework booklet.

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§3.3 **Probabilities Using Counting Methods (2 classes)**

Read *Goal* p. 151 text.

Outcomes:
1. Use the **fundamental counting principle** to determine probabilities. pp. 152-155
2. Use **combinations** to determine probabilities. pp. 152-155
3. Use **permutations** to determine probabilities. pp. 152-155

Recall that you can use **permutations** when **order is important** and **combinations** when **order is NOT important**.

E.g.: A committee of five people is selected from ten females and eight males.

a) What is the probability that there are exactly three females on the committee?

Order does NOT matter so we use combinations.

The total number of five-member committees is ${}_{18}C_5 = 8568$

The number of five-member committees with exactly three females (and two males) is ${}_{10}C_3 \times {}_8C_2 = 3360$.

So the probability that there are exactly three females on the committee will be

$$\frac{{}_{10}C_3 \times {}_8C_2}{{}_{18}C_5} = \frac{3360}{8568} = 39.2\%$$

b) What is the probability that there are exactly three males on the committee?

The total number of five-member committees is ${}_{18}C_5 = 8568$

The number of five-member committees with exactly three males (and two females) is ${}_8C_3 \times {}_{10}C_2 = 2520$.

So the probability that there are exactly three females on the committee will be

$$\frac{{}_8C_3 \times {}_{10}C_2}{{}_{18}C_5} = \frac{2520}{8568} = 29.4\%$$

c) What is the probability that there are exactly four females on the committee?

The total number of five-member committees is ${}_{18}C_5 = 8568$

The number of five-member committees with exactly four females (and one male) is ${}_{10}C_4 \times {}_8C_1 = 1680$.

So the probability that there are exactly three females on the committee will be

$$\frac{{}_{10}C_4 \times {}_8C_1}{{}_{18}C_5} = \frac{1680}{8568} = 19.6\%$$

E.g.: Steven, Brittany, Julie, and Max are volunteering along with five other students on their school's math team. All the students have equal ability. Determine the probability that Steven, Brittany, Julie, and Max will be chosen to fill the five spots on the team.

Order does NOT matter so we use combinations.

The total number of five-member teams is ${}_9C_5 = 126$

The number of five-member teams with Steven, Brittany, Julie, Max and one other is
${}_1C_1 \times {}_1C_1 \times {}_1C_1 \times {}_1C_1 \times {}_5C_1 = 5$ OR ${}_4C_4 \times {}_5C_1 = 5$.

So the probability that Steven, Brittany, Julie, Max and one other are on the team is

$$\frac{{}_1C_1 \times {}_1C_1 \times {}_1C_1 \times {}_1C_1 \times {}_5C_1}{{}_9C_5} = \frac{5}{126} \approx 4.0\%$$

E.g.: In the card game Crazy Eights, players are dealt 8 cards from a standard deck of 52 playing cards. Determine the probability that a hand will contain exactly 7 hearts.

Order does NOT matter so we use combinations.

The total number of eight-card hands is ${}_{52}C_8 = 752538150$

The number of eight-card hands with 7 hearts and 1 non heart is ${}_{13}C_7 \times {}_{39}C_1 = 66924$

So the probability that a hand will contain exactly 7 hearts is

$$\frac{{}_{13}C_7 \times {}_{39}C_1}{{}_{52}C_8} = \frac{66924}{752538150} = 0.00008893103958\ldots \approx 0.0089\%$$

E.g.: Access to a particular online site is password protected. Every member must create a password that consists of 3 capital letters followed by 2 digits. For each condition below, determine the probability that a password chosen at random will contain the letters A, B, and C.

a) Repetitions are not allowed in a password.

$$\frac{3 \times 2 \times 1 \times 10 \times 9}{26 \times 25 \times 24 \times 10 \times 9} = \frac{540}{1404000} = 0.0003846153846 = 0.03846153846\%$$

OR

$$\frac{{}_3P_3 \times {}_{10}P_2}{{}_{26}P_3 \times {}_{10}P_2}$$

b) Repetitions are allowed in a password.

$$\frac{3\times 2\times 1\times 10\times 10}{26\times 26\times 26\times 10\times 10}=\frac{600}{1757600}=0.0003413746017=0.03413746017\%$$

Sample Exam Question

There are 12 girls and 8 boys on a student council. Determine the probability that sub-committee of 3 students has:

(i) 3 girls

$$\frac{{}_{12}C_3\times {}_{8}C_0}{{}_{20}C_3}=0.1929824561\approx 19.3\%$$

(ii) 2 girls and 1 boy

$$\frac{{}_{12}C_2\times {}_{8}C_1}{{}_{20}C_3}=0.4631578947\approx 46.3\%$$

(iii) at least one girl

1G & 2B's OR 2G's & 1B OR 3G's & 0B's

$$\frac{{}_{12}C_1\times {}_{8}C_2}{{}_{20}C_3}+\frac{{}_{12}C_2\times {}_{8}C_1}{{}_{20}C_3}+\frac{{}_{12}C_3\times {}_{8}C_0}{{}_{20}C_3}=\frac{{}_{12}C_1\times {}_{8}C_2+{}_{12}C_2\times {}_{8}C_1+{}_{12}C_3\times {}_{8}C_0}{{}_{20}C_3}=0.950877193\approx 95.1\%$$

Sample Exam Question

Using the digits 0 to 9 and the 26 letters of the alphabet, a 6 character password is created that must begin and end with a letter and have digits for the remaining characters.

(i) Determine the probability that the password starts and ends with a vowel if letters are not case sensitive and no repeating characters are allowed.

The total number of 6-character passwords beginning and ending with a letter and having digits for the remaining characters with no repetition is $26\times 10\times 9\times 8\times 7\times 25=3276000$.

The number of 6-character passwords that start and end with a vowel if letters are not case sensitive and no repeating characters are allowed is $5\times 10\times 9\times 8\times 7\times 4=100800$.

So the probability that the password starts and ends with a vowel if letters are not case sensitive and no repeating characters are allowed is $\frac{5\times 10\times 9\times 8\times 7\times 4}{26\times 10\times 9\times 8\times 7\times 25}=\frac{100800}{3276000}=0.0307692308\approx 3.1\%$

(ii) Indicate if the probability changes if the letters are case sensitive. Justify the answer.

If the letters are case sensitive (e.g.: A is different from a), then the probability that the password starts and ends with a vowel if letters are not case sensitive and no repeating characters are allowed changes to $\frac{10\times10\times9\times8\times7\times9}{52\times10\times9\times8\times7\times51}=\frac{453600}{13366080}=0.0339366516\approx3.4\%$. So the probability increases since $\frac{4}{25}<\frac{9}{51}$.

E.g.: There are 7 teachers and 3 administrators at a conference. Find the probability of three different prizes being awarded to teachers only.

Method 1: Direct Reasoning (3 teachers selected & no administrators selected)

$$\frac{{}_7C_3\times{}_3C_0}{{}_{10}C_3}=\frac{35}{120}$$

Method 2: Indirect Reasoning (all possible ways 3 prizes can be awarded - # prizes to 1 administrators - # prizes to 2 administrators - # prizes to 3 administrators)

$$\frac{{}_{10}C_3-\left({}_7C_2\times{}_3C_1\right)-\left({}_7C_1\times{}_3C_2\right)-\left({}_7C_0\times{}_3C_3\right)}{{}_{10}C_3}=\frac{35}{120}$$

E.g.: A 4-digit PIN number can begin with any digit, except zero, and the remaining digits have no restriction. If repeated digits are allowed, find the probability of the PIN code beginning with a number greater than 7 and ending with a 3.

The number of 4-digit PIN numbers is $9\times10\times10\times10=9000$

The number of 4-digit PIN numbers beginning with a number greater than 7 and ending with a 3 is $2\times10\times10\times1=200$

The probability of the PIN code beginning with a number greater than 7 and ending with a 3 is $\frac{200}{9000}=0.0\overline{2}=2.\overline{2}\%$.

E.g.: Mark, Abby and 5 other students are standing in a line. Determine the probability that Mark and Abby are standing together. Determine the probability Mark and Abby are not standing together.

The total number of ways that 7 students can line up is $7!=5040$

The number of lineups that have Mark and Abby standing together is $2!\times6!=1440$

So the probability that Mark and Abby are standing together is $\frac{1440}{5040}=0.\overline{285714}\approx28.6\%$

E.g.: A bookcase contains 6 different math books and 12 different biology books. If a student randomly selects two of these books, determine the probability they are both math OR both biology books.

The number of ways to select any 2 books from the 18 books is ${}_{18}C_2 = 153$.

The number of ways to select 2 math (& 0 biology) books OR 2 biology (& 0 math) books is ${}_6C_2 \times {}_{12}C_0 + {}_6C_0 \times {}_{12}C_2 = 15 + 66 = 81$.

So the probability they are both math OR both biology books is $\dfrac{{}_6C_2 \times {}_{12}C_0 + {}_6C_0 \times {}_{12}C_2}{{}_{18}C_2} = \dfrac{81}{153} \approx 0.53$

E.g.: A jar contains 5 red, 7 blue, 5 purple and 3 yellow candies. If the total number of candies is 20, determine the probability that a handful of four candies contains one of each colour.

The number of ways to select any 4 candy from 20 candies is ${}_{20}C_4 = 4845$.

The number of ways to select one candy of each color is ${}_5C_1 \times {}_7C_1 \times {}_5C_1 \times {}_3C_1 = 525$

So the probability that a handful of four candies contains one of each colour is $\dfrac{{}_5C_1 \times {}_7C_1 \times {}_5C_1 \times {}_3C_1}{{}_{20}C_4} = \dfrac{525}{4845} \approx 0.11$

E.g.: Dar spells out COOKBOOK with letter tiles. The tiles are face down and mixed up. He asks Devon to arrange the tile in a row and turn them face up. If the row of tiles spells COOKBOOK, Devon will win a series of gift cards for local restaurants. Determine the probability that Devon will win.

The number of ways to arrange the letters of COOKBOOK is $\dfrac{8!}{4! \times 2!} = 840$.

There is only 1 way to spell COOKBOOK.

So the probability that Devon will win is $\dfrac{1}{840} \approx 0.0012$

Do #'s 1, 2, 4, 5, 10, 15, CYU/Practising pp.159-161 text in your homework booklet.

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§3.4 **Mutually Exclusive Events (2 classes)**

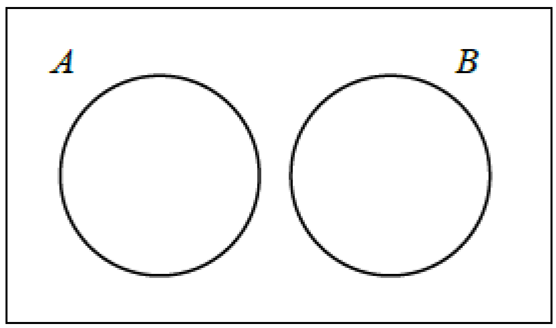
Read *Goal* p. 166 text.

Outcomes:
1. Define and give examples of **mutually exclusive** events. pp. 166, 175,698
2. Define and give examples of **non-mutually exclusive** events. pp. 168, 175
3. Solve problems involving mutually exclusive and non-mutually exclusive events. pp. 167-174

Def$^{\underline{n}}$: Events are **mutually exclusive** if the events CANNOT occur at the same time.

E.g.: The sun setting and the sun rising are mutually exclusive events.
E.g.: Turning right and turning left are mutually exclusive events.
E.g.: Rolling an even number on a die and rolling an odd number on a die are mutually exclusive events.

E.g.: Consider the experiment of rolling a six-sided die. Let the event *A* be "an even number is thrown" and the event *B* be "an odd number is thrown". Let's represent this situation using a Venn diagram.



List the outcomes for:

a) Event A _______________

b) Event B ________________

c) Event **A or B** ____

d) Event **A and B** ______

Complete the following:

a) $n(A) =$ ____
b) $n(B) =$ ____
c) $n(A \text{ or } B) =$ ____
d) $n(A \text{ and } B) =$ ____

Determine the following probabilities:

a) $P(A) = \_\_\_\_$

b) $P(B) = \_\_\_\_$

c) $P(A \text{ or } B) = \_\_\_\_$

d) $P(A \text{ and } B) = \_\_\_\_$

Note that events A and B have **no** common outcomes.

Notice that in the Venn diagram, the circle for set A and the circle for set B do NOT overlap. They are disjoint sets.

Def$^{\underline{n}}$: Events that have no common outcomes are called **mutually exclusive events**.

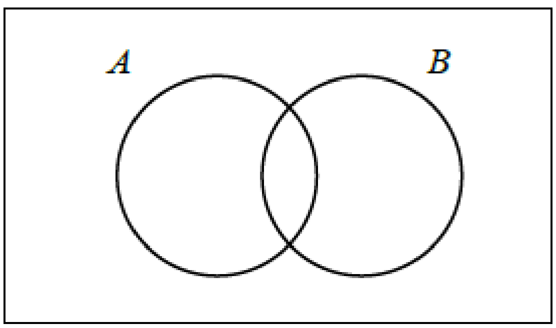
For mutually exclusive events:

****** $P(A \text{ or } B) = P(A) + P(B)$

OR

****** $P(A \cup B) = P(A) + P(B)$

E.g.: Consider the experiment of rolling a six-sided die. Let the event *A* be "an odd number is thrown" and the event *B* be "a multiple of three" is thrown. Let's represent this situation using a Venn diagram.



List the outcomes for:

a) Event A _______________

b) Event B ________________

c) Event **A or B** ____

d) Event **A and B** ______

Complete the following:

a) $n(A) =$ ____
b) $n(B) =$ ____
c) $n(A \text{ or } B) =$ ____
d) $n(A \text{ and } B) =$ ____

Determine the following probabilities:

a) $P(A) =$ ____
b) $P(B) =$ ____
c) $P(A \text{ or } B) =$ ____
d) $P(A \text{ and } B) =$ ____

Note that events A and B **have** common outcomes.

Notice that in the Venn diagram, the circle for set A and the circle for set B **do** overlap. They are non-disjoint sets.

Def$^{\underline{n}}$: Events that have common outcomes are called **non-mutually exclusive events**.

For non-mutually exclusive events:

****** $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

OR

****** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

An alternate formula is

****** $P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B)$

E.g.: Consider drawing a card from a standard deck. The following events are defined:

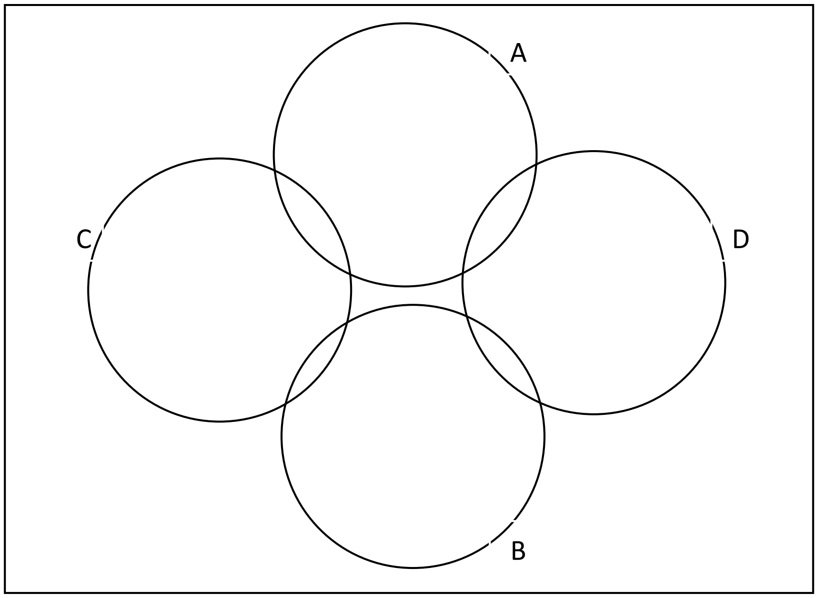
• Event *A* - a face card is selected
• Event *B* - an ace selected
• Event *C* - a heart is selected
• Event *D* - a black card is selected

a) State all the pairs of events which are mutually exclusive.

Event _____ and Event _____

Event _____ and Event _____

b) Use the Venn diagram to state all the pairs of events which are mutually exclusive.



c) Use the following information to determine if events A and B are mutually exclusive.

$$P(A)=\frac{1}{4};\quad P(B)=\frac{1}{3};\quad P(A \text{ or } B)=\frac{7}{12}$$

If events A and B are mutually exclusive then $P(A \text{ or } B)=P(A)+P(B)$.

$$P(A)+P(B)=\frac{1}{4}+\frac{1}{3}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}=P(A \text{ or } B)$$

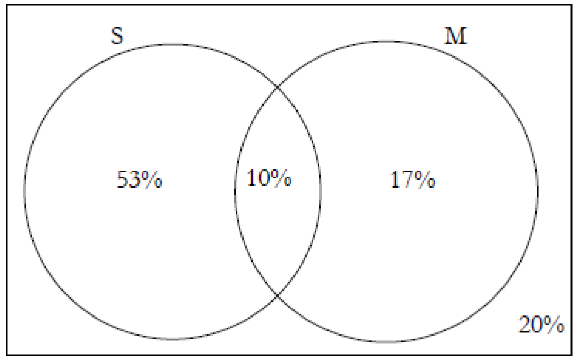
So events A and B are mutually exclusive.

E.g.: A class survey gave the following results.

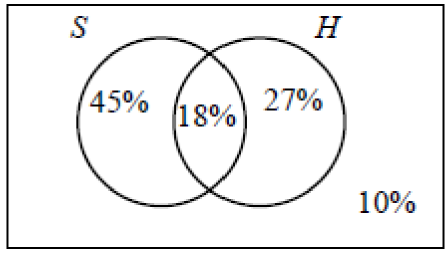
63% of students play sports.
27% of students play a musical instrument.
20% play neither sports nor a musical instrument.

Are playing sports and playing a musical instrument mutually exclusive events for the students in this class?

63% + 27% + 20% = 110%. This implies that 10% of the students in this class play sports and also play a musical instrument. Therefore the events are NOT mutually exclusive. See the Venn diagram below.



E.g.: A baseball team was surveyed to find out whether they stretched or hydrated before the game. The Venn diagram shows the percentage of players in each category.



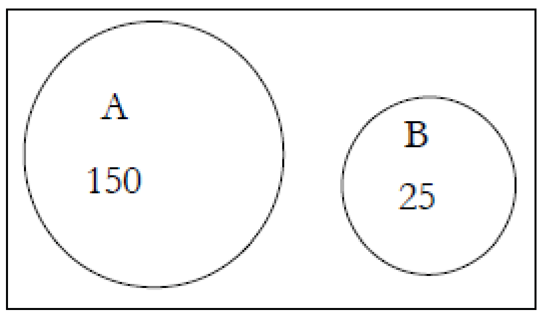
If a player is selected at random from the team, determine the probability that the student:

a) Stretched $P(S) = 45\% + 18\% = 63\% = 0.63$

b) Stretched and Hydrated $P(S \text{ and } H) = 18\% = 0.18$

c) Stretched or Hydrated $P(S \text{ or } H) = 45\% + 18\% + 27\% = 90\% = 0.9$

E.g.: If the sample space has 500 outcomes, find $P(\mathrm{A} \cup B)$ using the Venn diagram below.



Since events A and B are mutually exclusive,

$$P(\mathrm{A} \cup B) = P(A) + P(B) = \frac{150}{500} + \frac{25}{500} = \frac{175}{500} = 0.35$$

E.g.: The probability that Dana will make the hockey team is $\frac{2}{3}$. The probability that she will make the swimming team is $\frac{3}{4}$. If the probability of Dana making both teams is $\frac{1}{2}$, determine the probability that she will make:

a) at least one of the teams.

$$P(H) = \frac{2}{3}; \quad P(S) = \frac{3}{4}; \quad P(H \cap S) = \frac{1}{2}$$

Since $\frac{2}{3}+\frac{3}{4}>1$ making the hockey team and making the swim team are NOT mutually exclusive.

Therefore

$$P(H \cup S) = P(H) + P(S) - P(H \cap S)$$
$$P(H \cup S) = \frac{2}{3}+\frac{3}{4}-\frac{1}{2}=\frac{8+9-6}{12}=\frac{11}{12}=0.91\bar{6}$$

b) neither team.

$$P(H \cup S)' = 1 - P(H \cup S) = 1 - 0.91\bar{6} = 0.08\bar{3}$$

The probability that the Toronto Maple Leafs will win their next game is 0.5. The probability that the Montreal Canadiens will win their next game is 0.7. The probability that they will both win is 0.35. Determine the probability that one or the other will win (assume they don't play each other). Include a Venn diagram in your solution.

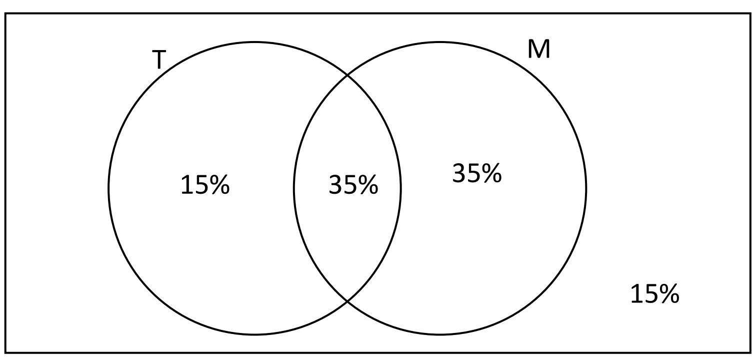
$$P(T) = 0.5; \quad P(M) = 0.7; \quad P(T \cap M) = 0.35$$

Since $0.5 + 0.7 > 1$ the events are NOT mutually exclusive.

Therefore

$$P(T \cup M) = P(T) + P(M) - P(T \cap M)$$
$$P(T \cup M) = 0.5 + 0.7 - 0.35 = 0.85$$

So the probability that one or the other will win is 0.85.



Do #'s 2, 3, 4, 7 a, 8 a, 13, 15, 19 CYU/Practising pp.176-180 text in your homework booklet.

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§3.5 **Conditional Probability (2 classes)**

Read *Goal* p. 182 text.

Outcomes:
1. Define and give examples of **independent** events. See notes
2. Define and give examples of **dependent** events. pp. 182, 696
3. Define **conditional probability**. pp. 182, 696
4. Solve problems involving dependent events. pp. 182-187

Def $^{\underline{n}}$: Events A and B are **independent** if the outcome event A does NOT affect the outcome of event B.

E.g.: Rolling a 2 on a die and then rolling a 6 on the die.
E.g.: Rolling a 5 on a six-sided die and tossing a coin and getting tails.
E.g.: Drawing cards from a deck **with replacement**.
E.g.: Drawing marbles from a bag **with replacement**.

If events are **independent**, $P(A \text{ and } B) = P(A) \times P(B)$

Def $^{\underline{n}}$: Events A and B are **dependent** if the outcome event A DOES affect the outcome of event B.

E.g.: Drawing cards from a deck **without replacement**.
E.g.: Drawing marbles from a bag **without replacement**.

If events are **dependent**, $P(A \text{ and } B) = P(A) \times P(B \mid A)$ OR $P(A \cap B) = P(A) \times P(B \mid A)$

The last expression in this formula, $P(B \mid A)$, is referred to as **conditional probability**. It is read as "the probability that event B occurs given that event A has already occurred."

Let's compare a situation with independent events with a similar situation with dependent events.

E.g.: **(Independent Events)** One card is drawn from a deck of cards **and is replaced**. A second card is then drawn. Consider the following events

A = {the first card is a spade}
B = {the second card is a spade}

Determine the probability that the first card drawn is a spade and the second card drawn is also a spade, $P(A \text{ and } B)$.

Since the events are independent because the cards are replaced, $P(A \text{ and } B) = P(A) \times P(B)$.

So $P(\text{spade and spade}) = \frac{13}{52} \times \frac{13}{52} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} = 0.0625$

E.g.: **(Dependent Events)** One card is drawn from a deck of cards **and is NOT replaced**. A second card is then drawn. Consider the following events

A = {the first card is a spade}
B = {the second card is a spade}

Determine the probability that the first card drawn is a spade and the second card drawn is also a spade, $P(A \text{ and } B)$.

Since the events are dependent because the cards are NOT replaced, $P(A \text{ and } B) = P(A) \times P(B \mid A)$.

So $P(\text{spade and spade}) = \frac{13}{52} \times \frac{12}{51} = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17} \approx 0.0588235294$

E.g.: A jar contains black and white marbles. Two marbles are chosen **without replacement**. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

$P(\text{Black and White}) = P\left(\text{B and W}\right) = 0.34; \quad P(\text{B}) = 0.47; \quad P(\text{W|B}) = ?$

Since the events are dependent, $P(B \text{ and W}) = P(B) \times P(W \mid B)$.

So,

$$0.34 = 0.47 \times P(W \mid B)$$
$$P(W \mid B) = \frac{0.34}{0.47} \approx 0.72$$

The probability of selecting a white marble on the second draw, given that the first marble drawn was black is about 0.72

E.g.: Mr. Maloney surveyed 100 students at Heritage Collegiate to determine if they were for or against having a barnyard theme for the graduation ceremony. The results are summarized in the table below.

	In Favour (I)	Against (A)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

Show that being male (M) and being in favour (I) of the barnyard theme are dependent events.

If being male and being in favour of the barnyard theme are **independent**, then $P(M \text{ and } I) = P(M) \times P(I)$. But,

$$P(M \text{ and } I) = \frac{15}{100} = 0.15 \text{ and } P(M) \times P(I) = \frac{60}{100} \times \frac{19}{100} = 0.144$$

So being male and being in favour of the barnyard theme are dependent events.

E.g.: A hockey team has jerseys in three different colours. There are 4 green, 6 white and 5 orange jerseys in the hockey bag. Todd and Blake are given a jersey at random. Tony, Sam, Lesley, and Dana were asked to write an expression representing the probability that both jerseys are the same colour. Which student correctly identified the probability and why?

Tony	$\left(\frac{2}{4}\right)\left(\frac{2}{6}\right)\left(\frac{2}{5}\right)$
Sam	$\left(\frac{2}{4}\right)+\left(\frac{2}{6}\right)+\left(\frac{2}{5}\right)$
Lesley	$\left(\frac{4}{15}\right)\left(\frac{3}{14}\right)+\left(\frac{6}{15}\right)\left(\frac{5}{14}\right)+\left(\frac{5}{15}\right)\left(\frac{4}{14}\right)$
Dana	$\left(\frac{4}{15}\right)\left(\frac{4}{15}\right)+\left(\frac{6}{15}\right)\left(\frac{6}{15}\right)+\left(\frac{5}{15}\right)\left(\frac{5}{15}\right)$

Both jerseys are the same colour if Todd is given a green jersey and Blake is given a green jersey OR if Todd is given a white jersey and Blake is given a white jersey OR if Todd is given an orange jersey and Blake is given an orange jersey. So,

$$P(\text{same colour}) = P(G) \times P(G \mid G) \text{ or } P(W) \times P(W \mid W) \text{ or } P(O) \times P(O \mid O)$$

$$= \frac{4}{15} \times \frac{3}{14} + \frac{6}{15} \times \frac{5}{14} + \frac{5}{15} \times \frac{4}{14}$$

So Lesley correctly identified the probability.

Do #'s 1-3, 4 a, 5, 7, 17 (see Example 1, pp. 182-83), CYU/Practising pp.188-191 text in your homework booklet.

§3.6 **Independent Events (2 classes)**

Summary for the Probability of Two Events A and B

1. If the events *A*, *B*, are **mutually exclusive**, then

$P(A \text{ or } B) = P(A) + P(B)$ OR $P(A \cup B) = P(A) + P(B)$

2. If the events *A*, *B*, are **NOT mutually exclusive**, then

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ OR $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3. If the events *A*, *B*, are **independent**, then

$P(A \text{ and } B) = P(A) \times P(B)$ OR $P(A \cap B) = P(A) \times P(B)$

4. If the events *A*, *B*, are **dependent**, then

$P(A \text{ and } B) = P(A) \times P(B | A)$ OR $P(A \cap B) = P(A) \times P(B | A)$

E.g.: Jane encounters 2 traffic lights on her way to school. There is a 55% chance that she will encounter a red light at the first light, and a 40% chance she will encounter a red light on the second light. The two traffic lights operate on separate timers. Determine the probability that both lights will be red on her way to school.

$P(\text{Red and Red}) = P(\text{R}_1 \text{ and } \text{R}_2) = ?; \quad P(\text{R}_1) = 0.55; \quad P(\text{R}_2) = 0.40$

These events are independent since the colour of the first light has no effect on the colour of the second light. Therefore,

$P(R_1 \text{ and } R_2) = P(R_1) \times P(R_2) = 0.55 \times 0.40 = 0.22$

E.g.: Two hockey players, Sydney and Phil, each independently take a penalty shot. Sydney has a $\frac{8}{10}$ chance of scoring, and Phil has a $\frac{3}{5}$ chance of scoring. What is the probability that:

a) both score?

The events are independent so $P(\text{score and score}) = \frac{8}{10} \times \frac{3}{5} = \frac{24}{50} = 0.48$

b) both miss?

$P(\text{miss and miss}) = \frac{2}{10} \times \frac{2}{5} = \frac{4}{50} = 0.08$

c) only one of them scores?

$$P(\text{score and miss}) \text{ OR } P(\text{miss and score}) = \left(\frac{8}{10}\times\frac{2}{5}\right)+\left(\frac{2}{10}\times\frac{3}{5}\right)=\frac{16}{50}+\frac{6}{50}=\frac{22}{50}=0.44$$

d) at least one of them scores?

Method 1: Direct Reasoning

$$P(\text{score and miss}) \text{ OR } P(\text{miss and score}) \text{ OR } P(\text{score and score})$$
$$=\left(\frac{8}{10}\times\frac{2}{5}\right)+\left(\frac{2}{10}\times\frac{3}{5}\right)+\left(\frac{8}{10}\times\frac{3}{5}\right)=\frac{16}{50}+\frac{6}{50}+\frac{24}{50}=\frac{46}{50}=0.92$$

Method 2: Indirect Reasoning

$$P(\text{at least one score})=1-P(\text{miss and miss})=1-\left(\frac{2}{10}\times\frac{2}{5}\right)=1-\frac{4}{50}=\frac{50}{50}-\frac{4}{50}=\frac{46}{50}=0.92$$

E.g.: The score at the end of regular time was Toronto 2 Montreal 2. After three shootout shots for each team in the first round, the game was still tied. The game will now be decided by a sudden death shootout where each team takes alternate shots on goal. Each team shoots once in each round. If both teams score or both teams miss, they go on to another round. From past records the probability that Toronto will score on a shootout shot is 0.7 and the probability that Montreal will score on a shootout shot is 0.6. What is the probability that:

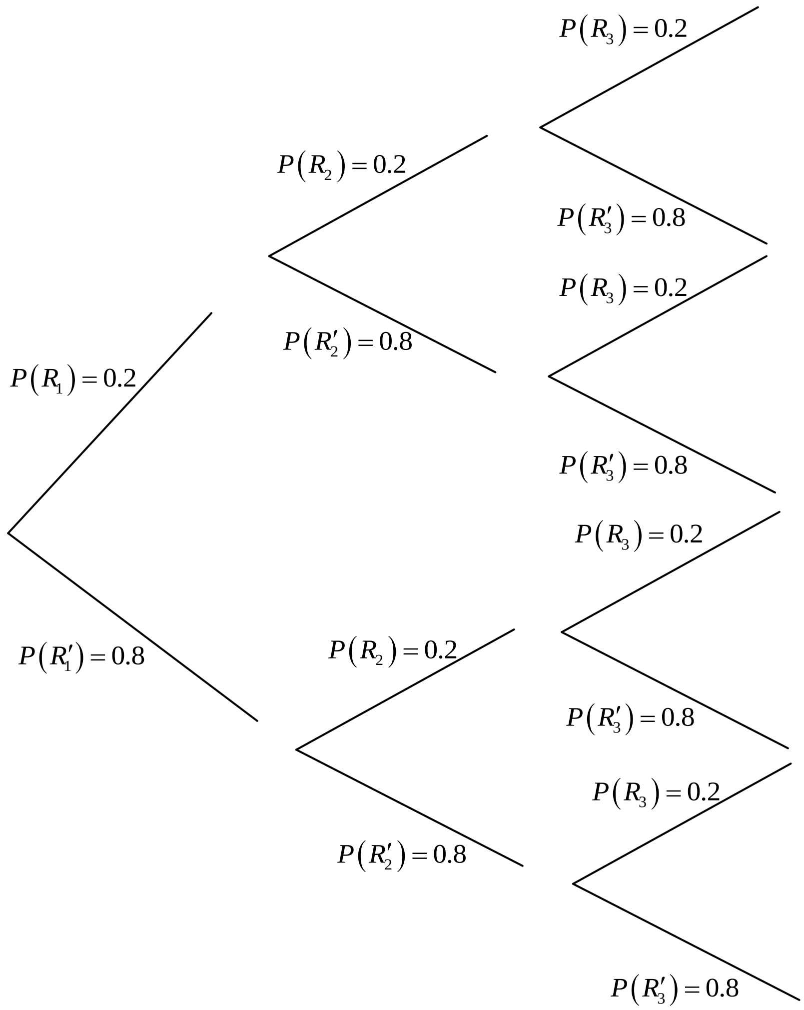
a) Toronto wins in the second round

$$P(\text{T}_{1\text{ score}} \text{ and } \text{M}_{1\text{ score}}) \text{ and } P(\text{T}_{2\text{ score}} \text{ and } \text{M}_{2\text{ not score}}) \text{ OR } P(\text{T}_{1\text{ not score}} \text{ and } \text{M}_{1\text{ not score}}) \text{ and } P(\text{T}_{2\text{ score}} \text{ and } \text{M}_{2\text{ not score}})$$
$$=P(\text{T}_{1\text{ score}})\times P(\text{M}_{1\text{ score}})\times P(\text{T}_{2\text{ score}})\times P(\text{M}_{1\text{ not score}})+P(\text{T}_{1\text{ not score}})\times P(\text{M}_{1\text{ not score}})\times P(\text{T}_{2\text{ score}})\times P(\text{M}_{1\text{ not score}})$$
$$=(0.7\times0.6)\times(0.7\times0.4)+(0.3\times0.4)\times(0.7\times0.4)=0.42\times0.28+0.12\times0.28=0.1512$$

b) Montreal wins in the second round?

$$P(\text{T}_{1\text{ score}} \text{ and } \text{M}_{1\text{ score}}) \text{ and } P(\text{T}_{2\text{ not score}} \text{ and } \text{M}_{2\text{ score}}) \text{ OR } P(\text{T}_{1\text{ not score}} \text{ and } \text{M}_{1\text{ not score}}) \text{ and } P(\text{T}_{2\text{ not score}} \text{ and } \text{M}_{2\text{ score}})$$
$$=P(\text{T}_{1\text{ score}})\times P(\text{M}_{1\text{ score}})\times P(\text{T}_{2\text{ not score}})\times P(\text{M}_{1\text{ score}})+P(\text{T}_{1\text{ not score}})\times P(\text{M}_{1\text{ not score}})\times P(\text{T}_{2\text{ not score}})\times P(\text{M}_{1\text{ score}})$$
$$=(0.7\times0.6)\times(0.3\times0.6)+(0.3\times0.4)\times(0.3\times0.6)=0.42\times0.18+0.12\times0.18=0.0972$$

E.g.: Beatrice is planning a family BBQ during the up-coming holiday weekend (3 days). She is considering cancelling the BBQ if it rains. The probability of a rain is 0.2 on each of the 3 days and the weather on each day is independent of the weather on the other days. Use the tree diagram below and determine the probability that:



a) it will rain all 3 days.

$$P(R_1 \text{ and } R_2 \text{ and } R_3)=P(R_1)\times P(R_2)\times P(R_3)=0.2\times 0.2\times 0.2=0.008$$

b) there will be only 1 day of rain.

$$\begin{aligned}
&P(\text{rain on only 1 day})\\
&=P\left(R_1 \text{ and } R_2' \text{ and } R_3'\right) \text{ OR } P\left(R_1' \text{ and } R_2' \text{ and } R_3\right) \text{ OR } P\left(R_1' \text{ and } R_2 \text{ and } R_3'\right)\\
&=(0.2\times 0.8\times 0.8)+(0.8\times 0.8\times 0.2)+(0.8\times 0.2\times 0.8)\\
&=0.128+0.128+0.128=0.384
\end{aligned}$$

c) that at least 2 days will be dry.

$$P(\text{at least 2 dry days}) = P(\text{2 dry days}) \text{ or } P(\text{3 dry days})$$
$$= P(R_1' \text{ and } R_2' \text{ and } R_3) \text{ OR } P(R_1' \text{ and } R_2 \text{ and } R_3') \text{ OR } P(R_1 \text{ and } R_2' \text{ and } R_3') \text{ OR } P(R_1' \text{ and } R_2' \text{ and } R_3')$$
$$= (0.8\times0.8\times0.2) + (0.8\times0.2\times0.8) + (0.2\times0.8\times0.8) + (0.8\times0.8\times0.8)$$
$$= 0.128 + 0.128 + 0.128 + 0.512 = 0.896$$

E.g.: The table shows how the students in a large high school generally travel to school.

	Bus B	Car C	Other O	Total
Male, M	**350**	100	75	
Female, F	**300**	75	100	
Total				

a) How many students attended the high school? 1000

b) If a student is selected at random, determine the probability that:

i. the student is male.

$$P(M) = \frac{525}{1000} = 0.525$$

ii. the student travels by bus.

$$P(B) = \frac{650}{1000} = 0.650$$

iii. the student is male and travels by bus.

$$P(M \text{ and } B) = \frac{350}{1000} = 0.350$$

iv. a female student travels by bus.

$$P(F \text{ and } B) = \frac{300}{1000} = 0.300$$

v. a student who drives is male.

$$P(M \text{ and } C) = \frac{100}{1000} = 0.100$$

c) Are the events "the student is female" and "the student travels by bus" independent events?

No, since $P(F \text{ and } B) = \frac{300}{1000} = 0.300$ and $P(F)\times P(B) = \frac{475}{1000}\times\frac{650}{1000} = 0.30875$

Do #'s 1, 2, 5, 6, 8, 9, 15, CYU/Practising pp. 198-200 text in your homework booklet.
Do #'s 4-7, 9-11, 13, 16, 17, 19-21 Practising pp. 205-206 text in your homework booklet.