Math 3201 Notes
Chapter 2: Counting Methods

Learning Goals: See p. 63 text.

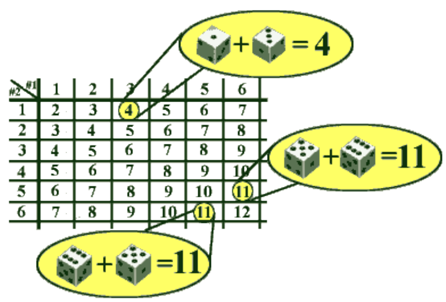
§2.1 **Counting Principles (2 classes)**

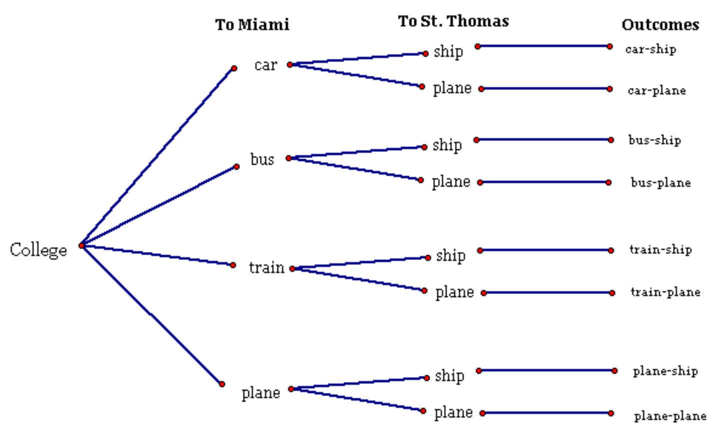
Outcomes:

1. Define the **sample space**. P. 66
2. Find the sample space by drawing a graphic organizer such as an **outcome table** or a **tree diagram**. pp. 66-67
3. Determine the number of outcomes in the sample space using a graphic organizer.
4. State the benefits and limitations in using graphic organizers to determine the sample space. p.67
5. State the **Fundamental Counting Principle**. p. 68
6. Use the Fundamental Counting Principle to determine the number of outcomes in the sample space. P. 68
7. State the benefits and limitations in using the Fundamental Counting Principle. p. 72

Def^{n}: The set of all possible outcomes is called the **sample space**. We can determine the sample space using graphic organizers such as lists, tables, and tree diagrams.

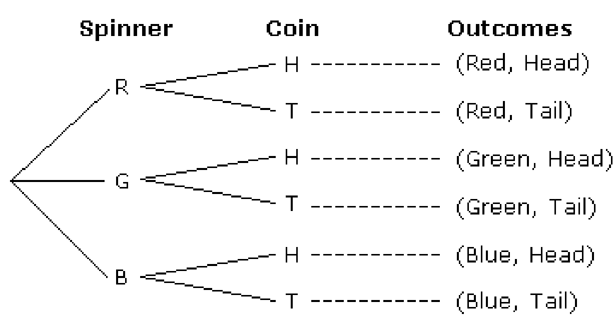
E.g.:





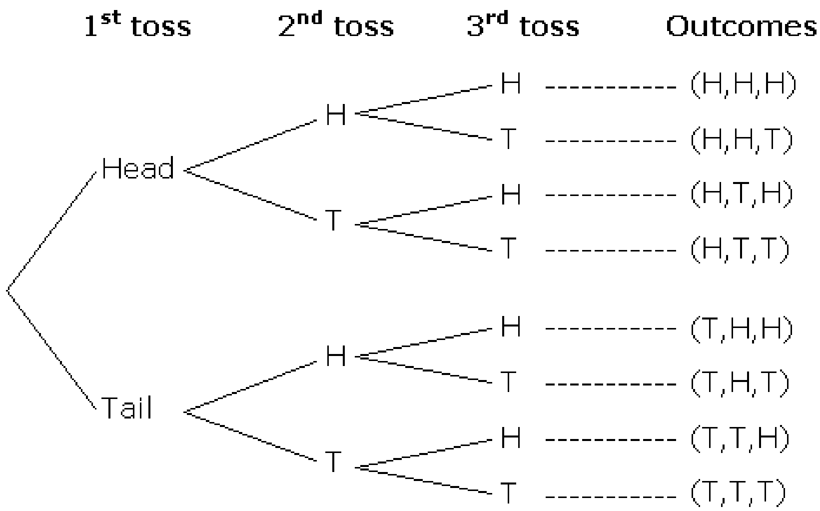
Often, we are interested not in the outcomes in the sample space but in the **number** of outcomes in the sample space.

E.g.: How many outcomes are in the sample space when the needle spins on a 3-color spinner and a coin is tossed?



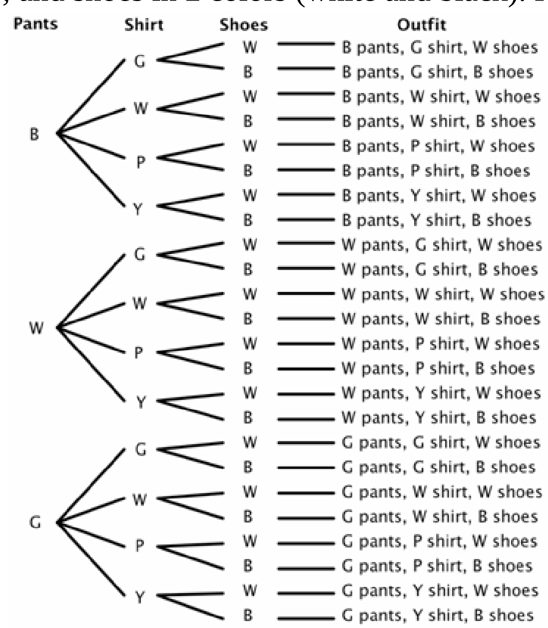
By drawing a tree diagram, we can determine that there are 6 possible outcomes.

E.g.: How many outcomes are in the sample space when we toss a coin 3 times?



By drawing a tree diagram, we can determine that there are 8 possible outcomes.

E.g.: You can choose from pants in 3 colors (black, white, green), shirts in 4 colors (green, white, purple, and yellow), and shoes in 2 colors (white and black). How many outfits can you make?



By drawing a tree diagram, we can determine that there are 24 possible outfits.

Notice that as the number of items we could choose from increased, the tree diagram became more difficult to draw. This shows one of the limitations of graphic organizers. There is another way that we can determine the number of outcomes in the sample space without having to draw some sort of diagram. It is called the **Fundamental Counting Principle**.

Spinner & Coin Example

# colors on spinner	# sides on coin	(# colors) X (# sides)	# outcomes from tree diagram
3	2	3 x 2 = 6	6

3-Coin Example

# sides on coin 1	# sides on coin 2	# sides on coin 3	(# sides) X (# sides) X (# sides)	# outcomes from tree diagram
2	2	2	2 x 2 x 2 = 8	8

Outfit Example

# colors pants	# colors shirt	# colors shoes	(# colors) X (# colors) X (# colors)	# outcomes from tree diagram
3	4	2	3 x 4 x 2 = 24	24

Make a conjecture based on the last three examples about how you can find the number of outcomes in the sample space.

Conjecture: ____

____

____

Fundamental Counting Principle

If there are a ways to perform one task and b ways of performing a second task, then there are $a \times b$ ways of performing task a AND then task b.

E.g.:

The Bagel Factory offers 12 different kinds of bagels and 4 types of cream cheese. How many possible combinations of bagels and cream cheese are there?

Solution:

12 kinds of bagels • **4** types of cream cheese = total outcomes

12 • 4 = 48

There are 48 different combinations of bagels and cream cheese.

E.g.:

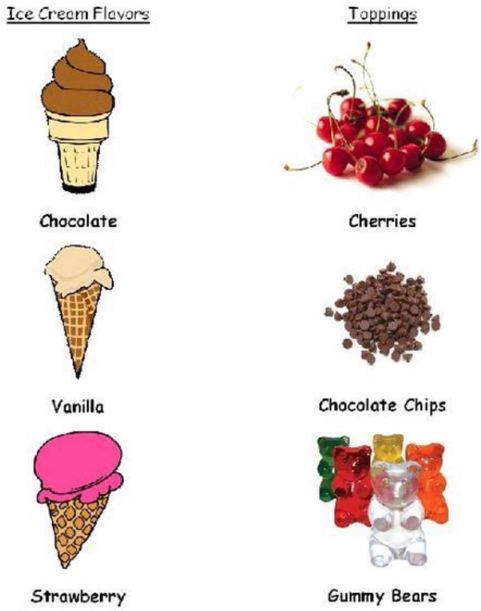
You have a photo that you want to mat and frame. You can choose from a blue, purple, red, or green mat and a metal or wood frame. Describe all of the ways you could frame this photo with one mat and one frame.

You can find all of the possible outcomes by making a tree diagram.

There should be 4 • 2 = 8 different ways to frame the photo.

E.g.:

Lucy and Andy are going to get ice cream. When they get there they see all their choices:



If Lucy and Andy can pick one flavor of ice cream and one topping, how many choices do they have?

E.g.:

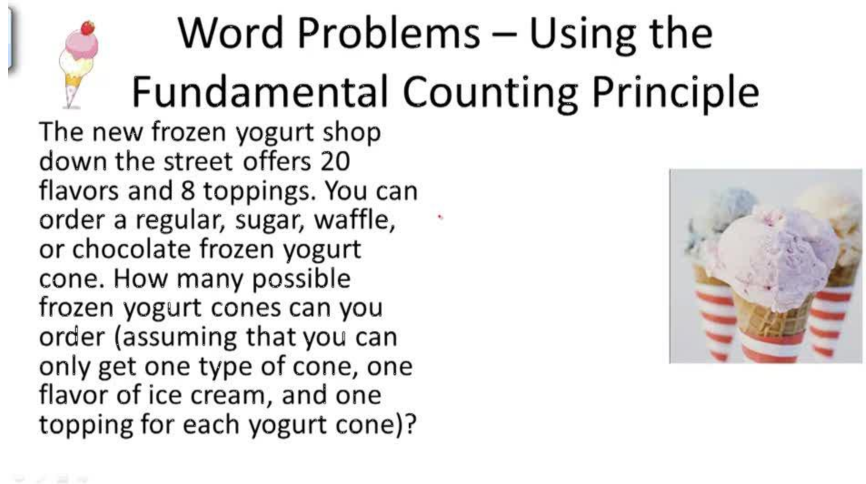
Jennifer has 3 skirts and 2 tops.
How many different outfits could she wear?

E.g.:

E.g.: How many 3-course meals can be made from the menu below?

E.g.:

E.g.:



Sample Exam Question

A Happy Meal consists of a burger, a side, and a drink. If you can choose from 6 different burgers, 2 different sides, and 8 different drinks, which expression indicates the number of possible different Happy Meals?

A) $6+2+8$
B) $6\times2+8$
C) $\dfrac{6\times8}{2}$
D) $6\times2\times8$

E.g.: How many combinations are possible for the lock to the right if each wheel contains the digits 0 to 9?

There are 10 choices for the first number, 10 for the second, and 10 for the third. According to the Fundamental Counting Principle, there are $10\times10\times10=10^3=1000$ possible lock combinations.

Your Turn: How many combinations are possible for the lock to the right if each wheel contains the digits 0 to 9?

Sometimes using the Fundamental Counting Principle is not quite so straightforward because there may be restrictions on some or all of the choices.

E.g.: In Newfoundland and Labrador, a license plate consists of 3 letters followed by 3 digits (e.g.: CXT 132).

a) How many license plate arrangements are possible?

There are 26 choices for the first letter, 26 choices for the second letter, 26 choices for the third letter, 10 choices for the first number, 10 choices for the second number, and 10 choices for the third number.

According to the Fundamental Counting Principle, there are $26\times26\times26\times10\times10\times10=17576000$ possible license plates.

b) How many license plate arrangements are possible if no letter and no digit can be repeated?

There are 26 choices for the first letter, 25 choices for the second letter, 24 choices for the third letter, 10 choices for the first number, 9 choices for the second number, and 8 choices for the third number.

According to the Fundamental Counting Principle, there are $26\times25\times24\times10\times9\times8=11232000$ possible license plates.

c) How many license plate arrangements are possible if repetition is allowed but vowels (A, E, I O, U) are not allowed?

There are 21 choices for the first letter, 21 choices for the second letter, 21 choices for the third letter, 10 choices for the first number, 10 choices for the second number, and 10 choices for the third number.

According to the Fundamental Counting Principle, there are $21\times21\times21\times10\times10\times10=9261000$ possible license plates.

E.g.: Canadian postal codes consist of a letter-digit-letter-digit-letter-digit arrangement.

a) How many postal codes are possible?

There are 26 choices for the first letter, 10 choices for the first digit, 26 choices for the second letter, 10 choices for the second digit, 26 choices for the third letter, and 10 choices for the last digit. According to the Fundamental Counting Principle, there are $26\times10\times26\times10\times26\times10=17576000$ possible postal codes.

b) In Newfoundland and Labrador, all postal codes begin with the letter A. How many Newfoundland and Labrador postal codes are possible?

There is 1 choice for the first letter, 10 choices for the first digit, 26 choices for the second letter, 10 choices for the second digit, 26 choices for the third letter, and 10 choices for the last digit. According to the Fundamental Counting Principle, there are $1\times10\times26\times10\times26\times10=676000$ possible postal codes for Newfoundland and Labrador.

E.g.: How many ways can you order the letters MUSIC if the first letter must be a consonant and the last letter must be a vowel.

There are 3 choices for the first letter and 2 choices for the last letter. There are 3 choices for the second letter, 2 choices for the third letter, and 1 choice for the fourth letter. According to the Fundamental Counting Principle, there are $3\times2\times3\times2\times1=36$ possible ways to arrange the letters.

Read "Key Ideas", p. 72 text.
Read "Need to Know", bullets 1 & 3, p. 72 text.
Do #'s 2, 5-7, 9, 11, 15, 16, CYU pp.73-75 in your homework booklet.

There are also situations where the Fundamental Counting Principle does NOT apply. Typically these situations involve the word "OR" instead of the word "AND".

E.g.: How many possible outcomes exist if you flip a coin AND then roll a die?

According to the Fundamental Counting Principle, there are $2\times6=12$ possible outcomes.

E.g.: How many possible outcomes exist if you either flip a coin OR roll a die?

There are 2 possible outcomes if you flip a coin (H,T). There are 6 possible outcomes if you roll a die (1-6), so there are $2+6=8$ possible outcomes. $\{H,T,1,2,3,4,5,6\}$

E.g.: When you roll a die, determine the number of ways you could get a number that is odd OR is greater than 4.

There are 3 numbers that are odd. There are 2 numbers that are greater than 4, so there are $3+2=5$ possible outcomes. $\{1,3,5,5,6\}$.

Sample Exam Question

A student travels to Bruce's Recreation to buy a new snowmobile. In the showroom there are 3 MXZ's 2 Renegades, 1 Summit, and 4 Skandics. Which expression illustrates the number of choices the student has?

A) $3+2+1+4$
B) $(3\times2)+(1\times4)$
C) $\dfrac{3\times2\times1}{4}$
D) $3\times2\times1\times4$

Read "Need to Know", bullet 2, p. 72 text.
Do #'s 3, 8, 14, CYU pp.73-74 in your homework booklet.

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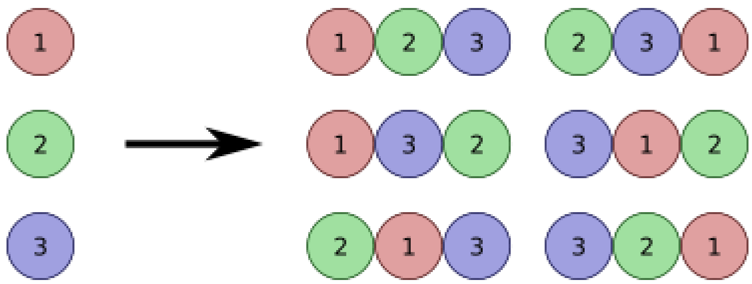
§2.2 **Introducing Permutations and Factorial Notation (1 class)**

Read "Goal" p. 76 text.

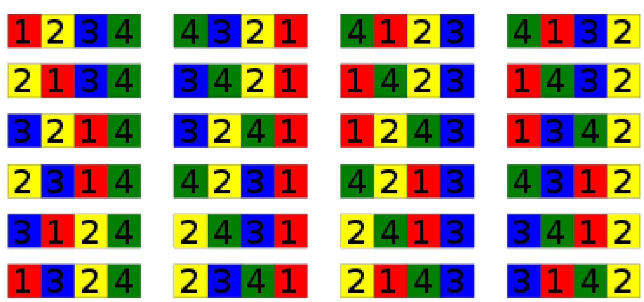
Outcomes:
1. Define a **permutation**. pp. 76, 698
2. Find the number of arrangements of *n* **different** objects taken *n* at a time. pp. 76, 77
3. Define **factorial notation**. pp. 77, 697
4. Simplify expression involving factorials. pp.78-79
5. Solve equations involving factorials. p. 80

Def^{n}: A **permutation** is an arrangement of different objects in a definite order. (ORDER MATTERS) If you change the order of the elements, you have a new permutation.

E.g.: The permutations of the numbers 1, 2, and 3 are below.



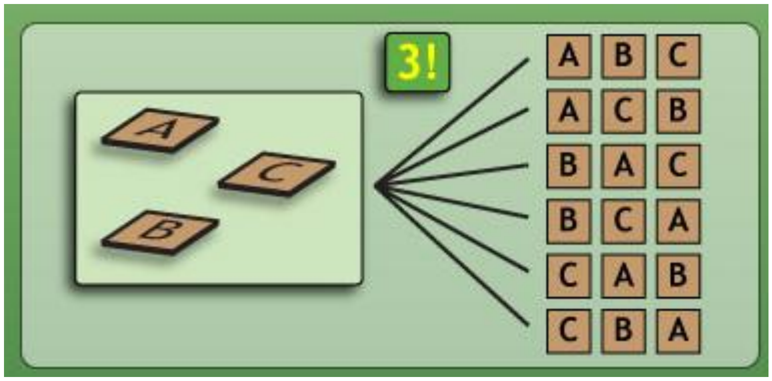
E.g.: The permutations of a red, a yellow, a blue, and a green block are below.



As in the last section, the graphic organizers can get time-consuming to draw as the number of elements increases. There is a way to determine the **number** of permutations without drawing a graphic organizer.

E.g.: Find the number of permutations of the letters A, B, and C.

There are 3 choices for the first letter, 2 choices for the second letter, and 1 choice last for the last letter. So there are $3\times2\times1=6$ permutations. (Check the diagram on the next page to see that 6 is correct.)



In mathematics there is a special way to write expressions such as $3\times2\times1$. We use factorial notation.

Def$^{\underline{n}}$: **Factorial notation** is a concise way to write the **product** of **consecutive, descending, natural** numbers. We use the "!" symbol to indicate factorial notation. Using factorial notation,

$$***n!=n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times3\times2\times1*** \quad n\in\mathbb{N}$$

E.g.: $3!=3\times2\times1$
E.g.: $4!=4\times3\times2\times1$
E.g.: $10!=10\times9\times8\times7\times6\times5\times4\times3\times2\times1$

Non e.g.: $4!\neq4\times5\times6\times7\times\cdots$
Non e.g.: $(-4)!\neq(-4)\times(-3)\times(-2)\times(-1)$

E.g.: How many ways can 7 students line up at the canteen counter?

There are 7 people who can be first in line, 6 people who can be second in line and so on. So the students can line up in $7!=7\times6\times5\times4\times3\times2\times1=5040$ ways.

In this chapter you will have to simplify expressions and solve equations involving factorials. Let's do some examples of simplifying expressions involving factorials.

E.g.: Simplify $\frac{8!}{4!3!}$

Expanding gives $\frac{8!}{4!3!}=\frac{8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times3\times2\times1}$

Canceling gives $\frac{8!}{4!3!}=\frac{8\times7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}\times3\times2\times1}=\frac{8\times7\times6\times5}{3\times2\times1}=\frac{8\times7\times\cancel{6}\times5}{\cancel{3\times2}\times1}=280$

Your Turn

Simplify $\frac{9!}{6!2!}$.

Do #'s 1 a, d, f, 3 a, b, c, 4, 5 b, c, d, pp. 81-82 text in your homework booklet.
Do #'s 7, 8, 12-14, pp. 82-83 text in your homework booklet.

E.g.: Simplify $\frac{(n+2)!}{(n)!}$

Expanding and cancelling gives

$$\frac{(n+2)!}{(n)!}=\frac{(n+2)\times(n+1)\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}$$

$=(n+2)(n+1)$

$=n^2+1n+2n+2$

$=n^2+3n+2$

Recall that factorials only apply to nonnegative integers (i.e. whole numbers). So in the last example, $n+2\geq 0$ and $n\geq 0$ OR $n\geq -2$ and $n\geq 0$. This means that $\frac{(n+2)!}{(n)!}$ is only defined for $n\geq 0$.

E.g.: Simplify $\frac{(n+4)!}{(n+2)!}$

Expanding and cancelling gives

$$\frac{(n+4)!}{(n+2)!}=\frac{(n+4)\times(n+3)\times\cancel{(n+2)}\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n+2)}\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}$$

$=(n+4)(n+3)$

$=n^2+3n+4n+12$

$=n^2+7n+12$

In this example, $n+4\geq 0$ and $n+2\geq 0$ OR $n\geq -4$ and $n\geq -2$. This means that $\frac{(n+4)!}{(n+2)!}$ is only defined for $n\geq -2$.

E.g.: Simplify $\frac{(n-1)!}{(n+1)!}$

Expanding and cancelling gives

$$\frac{(n-1)!}{(n+1)!}=\frac{\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{(n+1)\times(n)\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}$$

$=\frac{1}{(n+1)\times(n)}$

$\frac{1}{n^2+n}$

In this example, $n-1\ge 0$ and $n+1\ge 0$ OR $n\ge 1$ and $n\ge -1$. This means that $\frac{(n-1)!}{(n+1)!}$ is only defined for $n\ge 1$.

Do # 6 a, c, d, e, p. 82; #11 a, c, d, p. 94 text in your homework booklet.

You will also be required to solve equations involving factorials. Let's do some examples.

E.g.: Solve $\frac{(n+2)!}{(n+1)!}=10$

Expanding, cancelling, and solving gives

$$\frac{(n+2)\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}=10$$

$$n+2=10$$
$$n=8$$

Sample Exam Question

Solve $\frac{(n+5)!}{(n+3)!}=56$

Expanding, cancelling, and solving gives

$$\frac{(n+5)\times(n+4)\times\cancel{(n+3)}\times\cancel{(n+2)}\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n+3)}\times\cancel{(n+2)}\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}=56$$

$$(n+5)(n+4)=56$$
$$n^2+4n+5n+20=56$$
$$n^2+9n-36=0$$
$$(n+12)(n-3)=0$$
$$ZPP$$
$$n+12=0 \text{ or } n-3=0$$
$$\cancel{n=-12} \quad \text{ or } n=3$$
$$n=3$$

E.g.: Solve $\frac{(n)!}{(n-2)!}=182$

Expanding, cancelling, and solving gives

$$\frac{(n)\times(n-1)\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}=182$$

$(n)(n-1)=182$

$n^2-n=182$

$n^2-n-182=0$

$(n-14)(n+13)=0$

ZPP

$n-14=0$ or $n+13=0$

$n=14$ or $\cancel{n=-13}$

$n=14$

E.g.: Solve $\dfrac{2(n+3)!}{(n+1)!}=180$

First we divide each side by 2 to get

$$\frac{\frac{\cancel{2}(n+3)!}{(n+1)!}}{\cancel{2}}=\frac{180}{2}$$

$$\frac{(n+3)!}{(n+1)!}=90$$

Expanding, cancelling, and solving gives

$$\frac{(n+3)\times(n+2)\times\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n+1)}\times\cancel{(n)}\times\cancel{(n-1)}\times\cancel{(n-2)}\times\cancel{(n-3)}\times\ldots\times\cancel{3}\times\cancel{2}\times\cancel{1}}=90$$

$(n+3)(n+2)=90$

$n^2+2n+3n+6=90$

$n^2+5n+6=90$

$n^2+5n-84=$

$(n-7)(n+12)=0$

ZPP

$n-7=0$ or $n+12=0$

$n=7$ or $\cancel{n=-12}$

$n=7$

Do # 11, p. 82 text in your homework booklet.

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§2.3 **Permutations when all Objects are Different (2 classes)**

Read "Goal" p. 84 text.

Outcomes:

1. Find the number of arrangements of *n* different objects taken *r* at a time where $0 \le r \le n$. pp. 84-86
2. Give the formula used to find the number of permutations of *n* different objects taken *r* at a time where $0 \le r \le n$. p. 86

In our work with permutations so far, we have always used all the elements when making the permutations. Now let's look at permutations where we do NOT use all the elements.

E.g.: You have five flavours of ice cream – vanilla (V), chocolate (CH), strawberry (S), mint chip (MC), and cookie dough (CD). You want to choose three to mix into a bowl. How many different ways are there to make your bowl of ice cream?

There are 5 choices for the first flavour, 4 for the second, and 3 for the third choice.

Using the Fundamental Counting Principle, there are $5\times4\times3$ ways to make the bowl of ice cream.

Let's do some sneaky arithmetic with the expression $5\times4\times3$. We are going to multiply and divide this expression by the same expression.

$$(5\times4\times3)\times\frac{(2\times1)}{(2\times1)}$$
$$=\frac{5\times4\times3\times2\times1}{2\times1}$$

If we write this last expression using factorial notation, we get $\frac{5!}{2!}$

Do you see that we could also write this expression as

$$\frac{(\#\text{ flavours available})!}{(\#\text{ flavours available } - \#\text{ flavours chosen})!}=\frac{5!}{(5-3)!}?$$

E.g.: How many 4-digit numbers can be created from the digits 1, 3, 4, 6, 9, 7 and 2 without repeating any?

In this problem, you have 7 numbers and have to choose 4 of them.

There are 7 choices for the first digit, 6 choices for the second digit, 5 choices for the third digit, and 4 choices for the last digit.

Using the Fundamental Counting Principle, there are $7\times6\times5\times4$ possible 4-digit numbers.

Let's do some sneaky arithmetic with the expression $7\times6\times5\times4$. We are going to multiply and divide this expression by the same expression.

$$(7\times6\times5\times4)\times\frac{(3\times2\times1)}{(3\times2\times1)}$$
$$=\frac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1}$$

If we write this last expression using factorial notation, we get $\frac{7!}{3!}$

Do you see that we could also write this expression as

$$\frac{(\#\text{ digits available})!}{(\#\text{ digits available }-\#\text{ digits chosen})!}=\frac{7!}{(7-4)!}?$$

E.g.: How many 4-letter sequences can be made from the letters in the word HOMEY?

In this problem, you have 5 letters and have to choose 4 of them.

There are 5 choices for the first letter, 4 choices for the second letter, 3 choices for the third letter, and 2 choices for the last letter.

Using the Fundamental Counting Principle, there are $5\times4\times3\times2$ possible 4-letter sequences.

Let's do some sneaky arithmetic with the expression $5\times4\times3\times2$. We are going to multiply and divide this expression by the same expression.

$$(5\times4\times3\times2)\times\frac{(\times1)}{(\times1)}$$
$$=\frac{5\times4\times3\times2\times1}{1}$$

If we write this last expression using factorial notation, we get $\frac{5!}{1!}$

Do you see that we could also write this expression as

$$\frac{(\#\text{ letters available})!}{(\#\text{ letters available }-\#\text{ letters chosen})!}=\frac{5!}{(5-4)!}?$$

Make a conjecture about the formula which gives the number of permutations of *n* different objects taken *r* at a time.

Conjecture: The formula is ____.

Now let's introduce some notation. The number of permutations of *n* different objects taken *r* at a time is written as ${}_nP_r$.

Remember:
$n \longrightarrow$ total
$r \longrightarrow$ want

So

$$**** \; {}_nP_r = \frac{n!}{(n-r)!} \; ****$$

The Ice Cream Example: ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times\cancel{2\times1}}{\cancel{2\times1}} = 60$ possible bowls of ice cream.

The 4-digit Number Example: ${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{3}\times\cancel{2}\times\cancel{1}} = 840$ possible 4-digit numbers.

The 4-letter Sequence Example: ${}_5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = \frac{5\times4\times3\times2\times\cancel{1}}{\cancel{1}} = 120$ possible 4-letter sequences.

E.g.: In a 7-person race, how many different ways can the first 3 runners arrive at the finish line?

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 210 \text{ possible ways.}$$

E.g.: William has a collection of 8 toy train cars and creates a train using 4 of them. How many different ways could he have made the train?

$${}_8P_4 = \frac{8!}{(8-4)!} = \frac{8!}{4!} = \frac{8\times7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 1680 \text{ possible ways.}$$

Do #'s 1 b, d, e, 3, 5, 6, 8, p. 93 text in your homework booklet.

Sometimes you have to use the permutation formula to solve an equation. When you do this, keep in mind that factorial only applies to whole numbers. $\{0,1,2,3,4,5\ldots\}$.

E.g.: Solve ${}_nP_2 = 30$

Substituting into $\dfrac{n!}{(n-r)!}$ gives

$$\frac{n!}{(n-2)!} = 30$$

$$\frac{n\times(n-1)\times(n-2)\times(n-3)\times\ldots3\times2\times1}{(n-2)\times(n-3)\times\ldots3\times2\times1} = 30$$

$$n(n-1) = 30$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)(n+5) = 0$$

ZPP

$n-6=0$ or $n+5=0$

$n=6$ or $\cancel{n=-5}$

$n=6$

E.g.: Solve ${}_{n-1}P_2 = 12$

Substituting into $\dfrac{n!}{(n-r)!}$ gives

$$\frac{(n-1)!}{(n-1-2)!} = 12$$

$$\frac{(n-1)!}{(n-3)!} = 12$$

$$\frac{(n-1)\times(n-2)\times\cancel{(n-3)}\times\ldots\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{(n-3)}\times\ldots\cancel{3}\times\cancel{2}\times\cancel{1}} = 12$$

$$(n-1)(n-2) = 12$$

$$n^2 - 2n - 1n + 2 = 12$$

$$n^2 - 3n - 10 = 0$$

$$(n-5)(n+2) = 0$$

ZPP

$n-5=0$ or $n+2=0$

$n=5$ or $\cancel{n=-2}$

$n=5$

Your Turn: Solve ${}_nP_2 = 30$ Ans: 6

Your Turn Again: Solve ${}_{n+1}P_2 = 20$ Ans:4

Do # 11 (See *Need to Know*, bullet #2, p. 92 text), 15, p. 94 text in your homework booklet.
Do #'s 1, 3-6, 7 a, b, c, 8, 9 c, d, 10 a, p. 97 text.

§2.4 **Permutations when some Objects are Identical (2 classes)**

Read "Goal" p. 98 text.

Outcomes:

1. Find the number of arrangements of *n* objects taken *n* at a time when some of the objects are identical. pp. 98-103
2. Give the formula used to find the number of **different** arrangements of *n* objects taken *n* at a time when some of the objects are identical. P. 104

When some of the objects that you have to choose are identical, there will be fewer permutations because some outcomes will be repeated.

E.g.: Suppose you have 2 green marbles and 1 blue marble. How many **different** arrangements can you make?

We will use G_1, G_2, and B to represent the three marbles. According to the Fundamental Counting Principle, there are $3\times 2\times 1 = 3! = 6$ possible arrangements for the marbles. Let's list them in the table below.

G_1, G_2, B	G_1, B, G_2	B, G_1, G_2
G_2, G_1, B	G_2, B, G_1	B, G_2, G_1

You should notice that some of these arrangements would look identical. Let's remove the subscripts and cancel the duplicates.

G, G, B	G, B, G	B, G, G
~~G, G, B~~	~~G, B, G~~	~~B, G, G~~

The Fundamental Counting Principle indicates that there are 6 possible outcomes but in fact there are only 3 different outcomes because half of the outcomes are duplicates. So the number of **different** arrangements is $\frac{3!}{2!} = 3$. Where does the "2" come from in $\frac{3!}{2!}$?

E.g.: Suppose you have 3 green marbles and 1 blue marble. How many **different** arrangements can you make?

We will use G_1, G_2, G_3, and B to represent the four marbles. According to the Fundamental Counting Principle, there are $4\times 3\times 2\times 1 = 4! = 24$ possible arrangements for the marbles. Let's list them in the table below.

G_1, G_2, G_3, B	G_2, G_1, G_3, B	G_3, G_1, G_2, B	B, G_1, G_2, G_3
G_1, G_2, B, G_3	G_2, G_1, B, G_3	G_3, G_1, B, G_2	B, G_1, G_3, G_2
G_1, G_3, G_2, B	G_2, G_3, G_1, B	G_3, G_2, G_1, B	B, G_2, G_1, G_3
G_1, G_3, B, G_2	G_2, G_3, B, G_1	G_3, G_2, B, G_1	B, G_2, G_3, G_1
G_1, B, G_2, G_3	G_2, B, G_1, G_3	G_3, B, G_1, G_2	B, G_3, G_1, G_2
G_1, B, G_3, G_2	G_2, B, G_3, G_1	G_3, B, G_2, G_1	B, G_3, G_2, G_1

You should notice that some of these arrangements would look identical. Let's remove the subscripts and cancel the duplicates.

$\boxed{G, G, G, B}$	~~G, G, G, B~~	~~G, G, G, B~~	$\boxed{B, G, G, G}$
$\boxed{G, G, B, G}$	~~G, G, B, G~~	~~G, G, B, G~~	~~B, G, G, G~~
~~G, G, G, B~~	~~G, G, G, B~~	~~G, G, G, B~~	~~B, G, G, G~~
~~G, G, B, G~~	~~G, G, B, G~~	~~G, G, B, G~~	~~B, G, G, G~~
$\boxed{G, B, G, G}$	~~G, B, G, G~~	~~G, B, G, G~~	~~B, G, G, G~~
~~G, B, G, G~~	~~G, B, G, G~~	~~G, B, G, G~~	~~B, G, G, G~~

The Fundamental Counting Principle indicates that there are 24 possible outcomes but in fact there are only 4 different outcomes because some of the outcomes are duplicates. So the number of **different** arrangements is $\frac{4!}{3!} = 4$. Where does the "3" come from in $\frac{4!}{3!}$?

E.g.: Suppose you have 2 green marbles and 2 blue marbles. How many **different** arrangements can you make?

We will use G_1, G_2, B_1, and B_2 to represent the four marbles. According to the Fundamental Counting Principle, there are $4 \times 3 \times 2 \times 1 = 4! = 24$ possible arrangements for the marbles. Let's list them in the table below.

G_1, G_2, B_1, B_2	G_2, G_1, B_1, B_2	B_1, G_2, G_1, B_2	B_2, G_2, G_1, B_1
G_1, G_2, B_2, B_1	G_2, G_1, B_2, B_1	B_1, G_2, B_2, G_1	B_2, G_2, B_1, G_1
G_1, B_2, B_1, G_2	G_2, B_2, B_1, G_1	B_1, G_1, G_2, B_2	B_2, G_1, G_2, B_1
G_1, B_2, G_2, B_1	G_2, B_2, G_1, B_1	B_1, G_1, B_2, G_2	B_2, G_1, B_1, G_2
G_1, B_1, B_2, G_2	G_2, B_1, B_2, G_1	B_1, B_2, G_1, G_2	B_2, B_1, G_1, G_2
G_1, B_1, G_2, B_2	G_2, B_1, G_1, B_2	B_1, B_2, G_2, G_1	B_2, B_1, G_2, G_1

You should notice that some of these arrangements would look identical. Let's remove the subscripts and cancel the duplicates.

$\boxed{\text{G, G, B, B}}$	~~G, G, B, B~~	$\boxed{\text{B, G, G, B}}$	~~B, G, G, B~~
~~G, G, B, B~~	~~G, G, B, B~~	$\boxed{\text{B, G, B, G}}$	~~B, G, B, G~~
$\boxed{\text{G, B, B, G}}$	~~G, B, B, G~~	~~B, G, G, B~~	~~B, G, G, B~~
$\boxed{\text{G, B, G, B}}$	~~G, B, G, B~~	~~B, G, B, G~~	~~B, G, B, G~~
~~G, B, B, G~~	~~G, B, B, G~~	$\boxed{\text{B, B, G, G}}$	~~B, B, G, G~~
~~G, B, G, B~~	~~G, B, G, B~~	~~B, B, G, G~~	~~B, B, G, G~~

The Fundamental Counting Principle indicates that there are 24 possible outcomes but in fact there are only 6 different outcomes because some of the outcomes are duplicates. So the number of **different** arrangements is $\frac{4!}{(2!)(2!)} = 6$. Where does the “2” and the “2” come from in $\frac{4!}{(2!)(2!)}$?

---

2 green marbles and 1 blue marble example: $\frac{3!}{2!} = 3$ different arrangements.

3 green marbles and 1 blue marble example: $\frac{4!}{3!} = 4$ different arrangements.

2 green marbles and 2 blue marbles example: $\frac{4!}{(2!)(2!)} = 6$ different arrangements.

Do you see a pattern in the formula used to find the number of **different** arrangements? Make a conjecture about the formula that can be used to find the number of **different** permutations when some of the objects are identical.

Conjecture: The formula is ____.

E.g.: How many different arrangements can be made from the letters of the word HAPPY?

There are 5 letters and 2 P’s in HAPPY so there are $\frac{5!}{2!} = 60$ different arrangements.

E.g.: How many different arrangements can be made from the letters of the word BANANA?

There are 6 letters, 3 A’s and 2 N’s in BANANA so there are $\frac{6!}{(3!)(2!)} = 60$ different arrangements.

E.g.: How many distinct arrangements can be made from the letters of the word STATISTICS?

There are 10 letters, 3 S's, 3 T's, and 2 I's in STATISTICS so there are $\frac{10!}{(3!)(3!)(2!)} = 50400$ distinct arrangements.

Do #'s 2, 4, 5, 6 a, b, d, 15 a, pp. 104-107 text in your homework booklet.

Sometimes you have to find the number of different permutations when there are identical objects and there are also additional conditions on your choices.

E.g.: How many different ways can the letters of the word YAMAHA be arranged if the first letter must be an M and the last letter must be a Y?

There is only 1 choice for the first letter (M) and 1 choice for the last letter (Y). (M __ __ __ __ Y)

For the remaining 4 letters there 3 A's so there are are $\frac{4!}{3!} = 4$ choices.

Therefore, by the FCP there are $1 \times 4 \times 1 = 4$ different ways to arrange the letters (MAAAHY, MAAHAY, MAHAAY, MHAAY)

E.g.: How many different ways can the letters of the word YAMAHA be arranged if the first letter must be an A and the last letter must be a H?

There are 3 letters to choose from but they are identical so there are $\frac{3!}{3!} = 1$ choice for the first letter (A) and 1 choice for the last letter (H). For the remaining 4 letters there 2 A's so there are $\frac{4!}{2!} = 12$ choices.

Therefore, by the FCP there are $1 \times 12 \times 1 = 12$ different ways to arrange the letters. (AYAMAH, AYMAAH, AYAAMH, AMAYAH, AMYAAH, AMAAYH, AAMYAH, AAMAYH, AAYMAH, AAYAMH, AAAMYH, AAAYMH)

Do #'s 15 b, 16, 17, p. 107 text in your homework booklet.

E.g.: Sara has 3 identical Math 3201 text books (I don't know why!!), 2 identical History 3201 text books (I don't know why either!!), and 4 identical Physics 3204 text books (strange girl!!) to arrange on a shelf in her locker.

a) In how many different ways can she arrange the books on the shelf?

She can arrange the books in $\frac{9!}{(3!)(2!)(4!)} = 1260$ different ways.

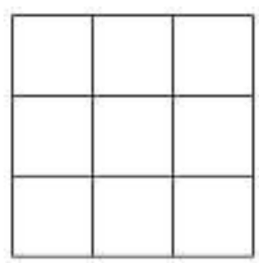
b) In how many different ways can she arrange the books on the shelf if books from the same subject must be grouped together?

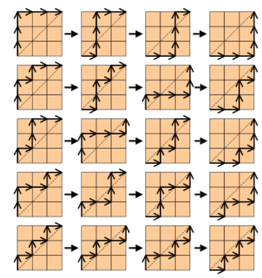
There are 3 different textbooks so there are $3! = 3 \times 2 \times 1 = 6$ different ways. (MMMHHPPPP, MMMPPPPHH, HHMMMPPPP, HHPPPPMMM, PPPPHHMMM, PPPPMMMHH)

Do # 7 a, p. 105 text in your homework booklet.

You can also apply the things we have learned to problems involving the number of routes from one point to another.

E.g.: How many different routes are there from the bottom left corner to the top right corner?

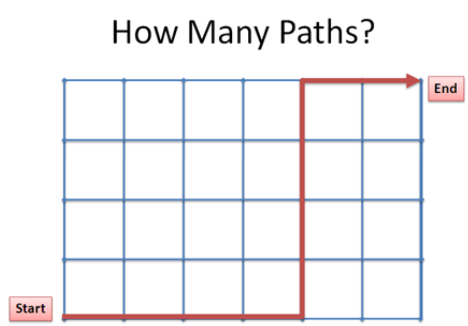




To get from the bottom left corner to the top right corner you must make 6 moves, 3 of which must be to the right and 3 of which must be up. So the number of different routes is essentially the same as the number of different arrangements of RRRUUU.

There are 6 letters, 3 of which are R's and 3 of which are U's. So there are $\frac{6!}{(3!)(3!)} = 20$ different routes.

E.g.:



To get from the **Start** to the **End** you must make 10 moves, 6 of which must be to the right and 4 of which must be up. So the number of different paths is essentially the same as the number of different arrangements of RRRRRRUUUU.

There are 10 letters, 6 of which are R's and 4 of which are U's. So there are $\frac{10!}{(6!)(4!)} = 210$ paths.

Do # 9, p. 105 text in your homework booklet.

§2.5 **Exploring Combinations (1 class)**

Read "Goal" p. 109 text.

Outcomes:

1. Define a **combination**. pp. 109, 696
2. Explain the difference between finding permutations and finding combinations. P. 109

The major difference between combinations and permutations is that with permutations, order matters, but with combinations, order does NOT matter.

Def^{n}: A **combination** is a grouping of objects where order does NOT matter. This means that rearranging the order of the elements does NOT make a new arrangement. The picture below contains ONE combination.

E.g. Sara (S), Jeremy (J), Megan (M), and Adam (A) all want to be on the music selection committee for this year's graduation ceremony. However, only 2 of them can serve on the committee. How many 2-member committees can be chosen from the 4 of them?

According to the FCP, there are $4 \times 3 = 12$ possible 2-member committees. Let's list them below.

SJ	JS	MS	AS
SA	JM	MJ	AJ
SM	JA	MA	AM

Note that many of these committees are duplicates. For example committee (Sara & Jeremy) is exactly the same as committee (Jeremy & Sara). Let's cancel the duplicates.

$\boxed{\text{SJ}}$	~~JS~~	~~MS~~	~~AS~~
$\boxed{\text{SA}}$	$\boxed{\text{JM}}$	~~MJ~~	~~AJ~~
$\boxed{\text{SM}}$	$\boxed{\text{JA}}$	$\boxed{\text{MA}}$	~~AM~~

The Fundamental Counting Principle indicates that there are 12 possible arrangements but in fact there are only 6 combinations because some of the outcomes are duplicates. So the number of combinations is $\frac{4!}{(2!)(2!)}=6$.

E.g. Sara (S), Jeremy (J), Megan (M), and Adam (A) all want to be on the music selection committee for this year's graduation ceremony. However, only 3 of them can serve on the committee. How many 3-member committees can be chosen from the 4 of them?

According to the FCP, there are $4\times3\times2=24$ possible 3-member committees. Let's list them below.

SJM	JSM	MSJ	ASJ
SJA	JSA	MSA	ASM
SMJ	JMS	MJS	AJS
SMA	JMA	MJA	AJM
SAJ	JAS	MAJ	AMS
SAM	JAM	MAS	AMJ

Note that a many of these committees are duplicates. For example committee (Sara, Jeremy, & Megan) is exactly the same as committee (Jeremy, Sara, & Megan) which is the same as committee (Megan, Sara, & Jeremy) and so on. Let's cancel the duplicates.

$\boxed{\text{SJM}}$	~~JSM~~	~~MSJ~~	~~ASJ~~
$\boxed{\text{SJA}}$	~~JSA~~	~~MSA~~	~~ASM~~
~~SMJ~~	~~JMS~~	~~MJS~~	~~AJS~~
$\boxed{\text{SMA}}$	$\boxed{\text{JMA}}$	~~MJA~~	~~AJM~~
~~SAJ~~	~~JAS~~	~~MAJ~~	~~AMS~~
~~SAM~~	~~JAM~~	~~MAS~~	~~AMJ~~

The Fundamental Counting Principle indicates that there are 24 possible arrangements but in fact there are only 4 combinations because some of the outcomes are duplicates. So the number of combinations is $\frac{4!}{(3!)(1!)}=4$.

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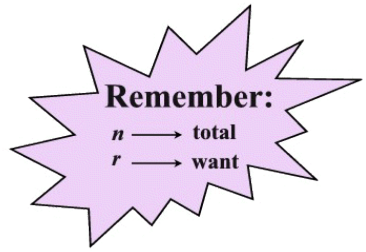
4 select 2 for committee example: # combinations is $\frac{4!}{(2!)(2!)}=6$

4 select 3 for committee example: # combinations is $\frac{4!}{(3!)(1!)}=4$

Do you see a pattern in the formula used to find the number of combinations? Make a conjecture about the formula that can be used to find the number of combinations.

Conjecture: The formula is ____. (See p. 112 text).

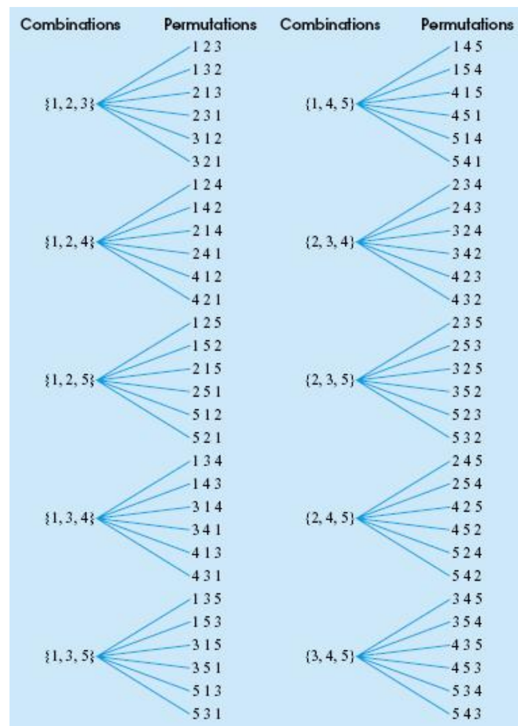
Now let's introduce some notation. The number of combinations of n different objects taken r at a time is written as ${}_nC_r$ or $\binom{n}{r}$.



So

$$****\ {}_nC_r = \frac{n!}{r!(n-r)!}\ ****$$

E.g.: 3 digits are to be selected from the #'s 1 to 5. Compare the number of combinations and the number of permutations.



There are ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$ permutations but only

${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = 10$ combinations.

Note that the number of combinations is less than the number of permutations.

Do #'s 2, 3, p. 110 text in your homework booklet.
Do # 13 a, p. 119 text in your homework booklet.

§2.6 **Combinations (2 classes)**

Read "Goal" p. 111 text.

Outcomes:

1. Solve combination problems. pp. 111-117

E.g.: Greco offers the following toppings for its pizzas: Pepperoni, Salami, Bacon Topping, Beef Topping, Sausage, Ham, Green Peppers, Onions, Mushrooms, Tomatoes, Black Olives, Hot Peppers & Pineapple. How many 3-topping pizzas can be made from this selection?

In this example you have 13 topping and you have to choose 3. Since order does not matter, there are $_{13}C_3 = \frac{13!}{3!(13-3)!} = \frac{13!}{3!10!} = 286$ three-topping pizzas.

E.g.: The volleyball team has 10 players, but only 6 can be on the court at one time. How many different ways can the team fill the court?

$_{10}C_6 = \frac{10!}{6!(10-6)!} = \frac{10!}{6!4!} = 210$ ways.

E.g.: There are 11 different marbles in a jar. How many different sets could you get by randomly picking 5 of them from the jar?

$_{11}C_5 = \frac{11!}{5!(11-5)!} = \frac{11!}{5!6!} = 462$ sets.

Do #'s 2, 3, 4 a, b, c, d, p. 118 text in your homework booklet.

E.g.: Four students from Heritage agree to share one 3-topping Greco pizza for lunch. They agree that the pizza must have green peppers but can have any two other toppings. How many pizzas could they order?

The first topping must be green pepper. There is only 1 way to select the first topping. For the remaining two toppings they can select from 12 options. Therefore they could order

$1 \times {}_{12}C_2 = 1 \times \frac{12!}{2!(12-2)!} = 1 \times \frac{12!}{2!10!} = 66$ pizzas.

Do # 8, p. 118 text in your homework booklet.

E.g.: A committee including 3 boys and 4 girls is to be formed from a group of 10 boys and 12 girls. How many different committees can be formed from the group?

There are 10 boys and 3 must be selected AND there are 12 girls and 4 must be selected. Therefore, there are ${}_{10}C_3 \times {}_{12}C_4 = 59400$ possible committees

Do # 10, p. 119 text in your homework booklet.

E.g.: How many 6-person graduation committees can be chosen from a group of 18 males and 22 females?

$$ {}_{40}C_6 = \frac{40!}{6!(40-6)!} = \frac{40!}{6!34!} = 3838380 $$

E.g.: How many 6-person graduation committees can be chosen from a group of 18 males and 22 females if there must be exactly 4 males?

${}_{18}C_4 \times {}_{22}C_2 = 706860$ committees.

E.g.: How many 6-person graduation committees can be chosen from a group of 18 males and 22 females if there can be no males?

${}_{22}C_6 = 74613$ committees.

E.g.: How many 6-person graduation committees can be chosen from a group of 18 males and 22 females if there must be at least 4 males?

There can be 4 males and 2 females OR 5 males and 1 female OR 6 males.

${}_{18}C_4 \times {}_{22}C_2 + {}_{18}C_5 \times {}_{22}C_1 + {}_{18}C_6 \times {}_{22}C_0 = 706860 + 188496 + 18564 = 913920$ committees.

Do # 11, p. 119 text in your homework booklet.

E.g.: There are 18 males and 22 females in Level III. A group of 7 students is needed for the decoration committee for the graduation. How many groups of 7 students with at least 2 females can be chosen?

Direct Reasoning:

There can be 2 females and 5 males OR 3 females and 4 males OR 4 females and 3 males OR 5 females and 2 males OR 6 females and 1 male OR 7 females and 0 males.

${}_{22}C_2 \times {}_{18}C_5 + {}_{22}C_3 \times {}_{18}C_4 + {}_{22}C_4 \times {}_{18}C_3 + {}_{22}C_5 \times {}_{18}C_2 + {}_{22}C_6 \times {}_{18}C_1 + {}_{22}C_7 \times {}_{18}C_0 =$
$1979208 + 4712400 + 5969040 + 4029102 + 1343034 + 170544 = 18203328$ committes.

Indirect Reasoning:

There are ${}_{40}C_7 = 18643560$ seven-member committees.

There are ${}_{22}C_1 \times {}_{18}C_6 = 408408$ committees with 1 female.

There are ${}_{22}C_0 \times {}_{18}C_7 = 31824$ committees with 0 females.

So there are 18643560 – 408408 – 31824 = 18203328 committees.

Do # 18, p. 120 text in your homework booklet.

E.g.: Solve ${}_nC_2 = 21$

Substituting into $\dfrac{n!}{r!(n-r)!}$ gives

$$\frac{n!}{2!(n-2)!} = 21$$

$$\frac{n\times(n-1)\times \cancel{(n-2)}\times\ldots \cancel{3\times2\times1}}{2\times \cancel{(n-2)}\times\ldots \cancel{3\times2\times1}} = 21$$

$$\frac{n(n-1)}{2} = 21$$

$$n(n-1) = 2(21)$$

$$n^2 - n - 42 = 0$$

$$(n-7)(n+6) = 0$$

ZPP

$n-7=0$ OR $n+6=0$

$n=7$ OR $\cancel{n=-6}$

$n=7$

E.g.: Solve ${}_{n+1}C_1 = 20$

Substituting into $\dfrac{n!}{r!(n-r)!}$ gives

$$\frac{(n+1)!}{1!(n+1-1)!} = 20$$

$$\frac{(n+1)!}{1 \times n!} = 20$$

$$\frac{(n+1)\,\cancel{(n)}\,\cancel{(n-1)}\,\cancel{(n-2)} \times \ldots \cancel{3 \times 2 \times 1}}{1 \times \cancel{n} \times \cancel{(n-1)}\,\cancel{(n-2)} \times \ldots \cancel{3 \times 2 \times 1}} = 20$$

$n + 1 = 20$

$n + 1 - 1 = 20 - 1$

$n = 19$

Do # 15 a, p. 119 text in your homework booklet.

§2.7 **Solving Counting Problems (2 classes)**

Read "Goal" p. 121 text.

Outcomes:

1. Determine if a problem involves permutations or combinations.
2. Solve permutation problems. pp. 121-128
3. Solve combination problems. pp. 121-128

To determine if a problem involves permutations or combinations you must determine whether or not order is important. If order is important (i.e.: if changing the order of the elements gives a new arrangement) then the problems involves permutations. If order is NOT important (i.e.: if changing the order of the elements does NOT give a new arrangement) then the problems involves combinations. After that, you need to determine if the problem is a simple permutation/combination problem or if it has repetition (e.g. repetition allowed/not allowed), conditions (e.g.: first letter is a vowel), or cases (e.g.: at least/at most).

Recall that sometimes the simplest way to solve the problem is to use the FCP.

Sample Exam Question

How many 4-digit password can be made using the numbers 0-9 if repetition of numbers is allowed and 0 cannot be used for the first digit.

There are 9 choices for the first digit, 10 for the second digit, 10 for the third digit, and 10 for the last digit.

According to the FCP, there are $9 \times 10 \times 10 \times 10 = 9000$ passwords.

Sample Exam Question

There are 10 defensemen and 14 forwards on a hockey team. Determine the number of ways in which a starting lineup of 5 players (no goalie) can be selected.

Order is NOT important in this problem so it involves combinations.

You have 10 defensemen and have to choose 2. This can be done in ${}_{10}C_2 = 45$ ways.
You have 14 forwards and have to choose 3. This can be done in ${}_{14}C_3 = 364$ ways.

So choosing 2 defensemen AND 3 forwards can be done in ${}_{10}C_2 \times {}_{14}C_3 = 16380$ ways.

Sample Exam Question

How many outfits can be made from 3 pairs of pants, 6 shirts and 2 pairs of shoes?

Order is NOT important in this problem so it involves combinations.

You have 3 pairs of pants and have to choose 1. This can be done in ${}_3C_1 = 3$ ways.
You have 6 shirts and have to choose 1. This can be done in ${}_6C_1 = 6$ ways.
You have 2 pairs of shoes and have to choose 1. This can be done in ${}_2C_1 = 2$ ways.

So choosing 1 pair of pants AND 1 shirt AND 1 pair of shoes can be done in ${}_3C_1 \times {}_6C_1 \times {}_2C_1 = 36$ ways. So 36 outfits are possible.

Note that this problem could also have been done using the FCP.

E.g.: In how many ways can 12 runners place first, second and third?

Order is important here so this is a permutation problem.

You have 12 runners and have to choose 3. This can be done in ${}_{12}P_3 = 1320$ ways.

Note that this problem could also have been done using the FCP.

E.g.: A cookie jar contains three chocolate chip, two peanut butter, one lemon, one almond and five raisin cookies.
a) In how many ways can you select 4 cookies?

Order does not matter so this is a combination problem.

You have 12 cookies and have to select 4. This can be done in ${}_{12}C_4 = 495$ ways.

b) In how many ways can you select 4 cookies if you select at least one chocolate chip cookie?

This means that you could have 1, 2, or 3 chocolate chip cookies out of the 4 cookies chosen.

Case 1: 1 CC and 3 non CC. This can be done in ${}_3C_1 \times {}_9C_3 = 252$ ways.

Case 2: 2 CC and 2 non CC. This can be done in ${}_3C_2 \times {}_9C_2 = 108$ ways.

Case 1: 3 CC and 1 non CC. This can be done in ${}_3C_1 \times {}_9C_3 = 9$ ways.

So the number of ways to select 1CC and 3 non CC OR 2 CC and 2 non CC OR 3 CC and 1 non CC is $252 + 108 + 9 = 369$ ways.

E.g.: Using the letters of the word **INTERMITTENT** how many 12-letter arrangements can be formed:

a) without restrictions.
b) if each word begins with **T** and ends with **N.**

a) There are 12 letters, 2 I's, 2 N's, 4 T's and 2 E's. So there are $\frac{12!}{2!\times 2!\times 4!\times 2!} = 2494800$ twelve-letter words.

b) There are 4 T's so there are ${}_4C_1 = 4$ ways to select the first letter.

There are 2 N's so there are ${}_2C_1 = 2$ ways to select the last letter.

If one T and one N are selected, 10 letters remain so there are $4\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1\times 2$ twelve-letter words possible.

However, some of these words are repeated because there are 2 I's, 4 T's 2 E's and 2 N's.

Therefore, there are $\frac{4\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1\times 2}{2!\times 2!\times 4!\times 2!} = 151200$ different twelve-letter words

Do #'s 1, 2, 4, 5, 6, 9, 11-13, 16, pp. 126-127 text in your homework booklet.
Do #'s 1-4, 8-15, 18 (triplets are identical), pp. 131-132 text in your homework booklet.