Section 6.3: Solving Exponential Equations

Exponent Laws	***Examples:***		
1. Zero Exponent: $b^0 = 1$	a) $\left(\frac{2}{3}\right)^0$ $= 1$	b) $5x^0$ $5(1) = 5$	c) $(5x^2y^3)^0$ $= 1$
2. Negative Exponent: $b^{-x} = \frac{1}{b^x}$	a) 3^{-2} $= \frac{1}{3^2}$ $= \frac{1}{9}$	b) $\left(\frac{3}{4}\right)^{-2}$ $= \left(\frac{4}{3}\right)^2$ $= \frac{16}{9}$	c) $\frac{1}{x^{-2}}$ $\frac{x^2}{1}$ $= x^2$

multiply

3. Product Rule:

$b^x \cdot b^y = b^{x+y}$

a) $2^2 \times 2^3$

$= 2^{2+3}$

$= 2^5$

b) $x^4 \cdot x^2$

$= x^{4+2}$

$= x^6$

c) $5x^2y^4(3x^3y^2)$

$= 15x^{2+3}y^{4+2}$

$= 15x^5y^6$

Exponent Laws ***Examples:***

Division

4. Quotient Rule:

$$\boxed{\frac{b^x}{b^y}=b^{x-y}}$$

a) $\frac{x^5}{x^3}$

$= x^{5-3}$

$= x^2$

b) $\frac{12x^2}{4x^{-3}}$

$= 3x^{2-(-3)}$

$= 3x^{2+3}$

$= 3x^5$

c) $\frac{16x^3y^7}{8x^5y^4}$

$= 2x^{3-5}y^{7-4}$

$= 2x^{-2}y^3$

$= 2\frac{1}{x^2}y^3$

$= \frac{2y^3}{x^2}$

Exponents

5. Power Rule:

$$\boxed{(b^x)^y = b^{xy}}$$

a) $(2^5)^2$

$= 2^{5\cdot 2}$

$= 2^{10}$

b) $(2x^{-2})^3$

$= 2^3x^{-2\cdot 3}$

$= 8x^{-6}$

$= 8\frac{1}{x^6} = \frac{8}{x^6}$

c) $\left(\frac{1}{3x^5}\right)^{-2}$

$= \frac{1}{3^{-2}x^{5\cdot -2}}$

$= \frac{1}{3^{-2}x^{-10}}$

$= 3^2x^{10} = 9x^{10}$

6. Rational Exponents:

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

a) $9^{\frac{1}{2}}$

$= \sqrt[2]{9}$

$= 3$

b) $64^{\frac{2}{3}}$

$= \sqrt[3]{64^2}$

$= 4^2$

$= 16$

c) $\left[\left(\frac{16}{9}\right)^{-3}\right]^{\frac{1}{2}}$

$= \left(\frac{16}{9}\right)^{-\frac{3}{2}}$

$= \left(\frac{9}{16}\right)^{\frac{3}{2}}$

$= \left(\sqrt[2]{\frac{9}{16}}\right)^3$

$= \left(\frac{3}{4}\right)^3$

$= \frac{27}{64}$

$\sqrt{9}$

$9^{\frac{1}{3}} = \sqrt[3]{9}$

7. Common Base Rule:

$\boxed{b^x = b^y \text{ if and only if } x = y}$

a) $2^x = 2^3$

$x = 3$

b) $5^{x+3} = 5^4$

$x + 3 = 4$
$x = 4 - 3$
$x = 1$

Example 1:

Express each of the following as a power with a base of 2.

a) 8

$= 2^3$

b) $\frac{1}{16}$

$= \frac{1}{2^4}$

$= 2^{-4}$

c) 8^{-2}

$(2^3)^{-2}$

$= 2^{3\cdot(-2)}$

$= 2^{-6}$

d) $8^{\frac{2}{3}}\left(\sqrt{16}\right)^3$

$= (2^3)^{2/3} \quad (16)^{3/2}$

$= 2^{3\cdot 2/3} \quad (2^4)^{3/2}$

$= 2^2 \cdot 2^{4\cdot 3/2}$

$= 2^2 \cdot 2^6 = 2^8$

Your Turn

Express each of the following as a power with a base of 3.

a) 27^2

$= (3^3)^2$

$= 3^{3\cdot 2}$

$= 3^6$

b) $\left(\frac{1}{9}\right)^4$

$= \left(\frac{1}{3^2}\right)^4$

$= (3^2)^{-4}$

$= 3^{2\cdot(-4)}$

$= 3^{-8}$

c) $27^{\frac{2}{3}}\left(\sqrt[3]{81}\right)^6$

$= (3^3)^{2/3} \quad \left(\sqrt[3]{3^4}\right)^6$

$= 3^{3\cdot 2/3} \quad \left(3^{\frac{4}{3}}\right)^6$

$= 3^2 \quad 3^{\frac{4}{3}\cdot 6}$

$= 3^2 \quad 3^8$

$= 3^{2+8} = 3^{10}$

To Solve Exponential Equations:

- write both sides of the equation with the same base
- equate the exponents

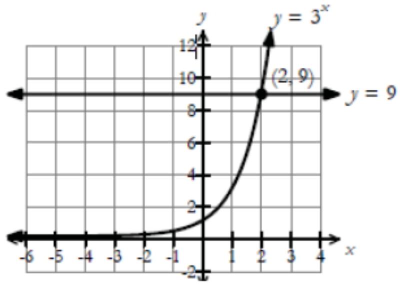
Example 2:

Solve for x: $3^x = 9$

$3^x = 3^2$

$x = 2$

The solution for the equation $3^x = 9$ can also be depicted graphically. We treat each side of the equation as 2 separate functions. The x-value of the point of intersection is the solution to the equation, $x = 2$.

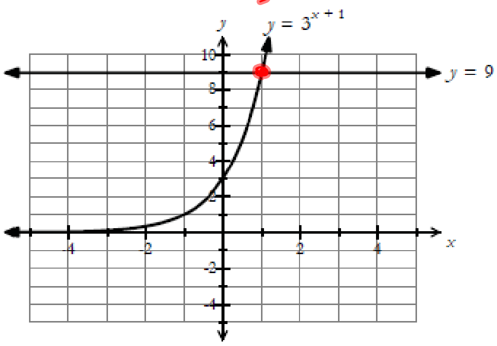


Example 3:

$(1, 9)$

a) Use the graph to determine the solution for $3^{x+1} = 9$.

$x = 1$



b) Verify the solution algebraically.

$3^{x+1} = 9$

$3^{x+1} = 3^{2}$

$x + 1 = 2$

$\boxed{x = 1}$

Example 4:

Solve each equation algebraically:

Rewrite each equation with the same base and equate the exponents.

a) $2^{x-1}=16$

$2^{x-1}=2^{4}$

$x-1=4$

$x=4+1$

$\boxed{x=5}$

b) $4^{2x}=8^{2x-3}$

$2^{2(2x)}=2^{3(2x-3)}$

$2(2x)=3(2x-3)$

$4x=6x-9$

$4x-6x=-9$

$\frac{-2x}{-2}=\frac{-9}{-2}$

$x=\frac{9}{2}$ or $4\frac{1}{2}$

c) $\frac{4(3^{x+2})}{4}=\frac{36}{4}$

$3^{x+2}=9$

$3^{x+2}=3^{2}$

$x+2=2$

$x=2-2$

$\boxed{x=0}$

d) $8^{3x-4}+7=71$

$8^{3x-4}=71-7$

$8^{3x-4}=64$

$8^{3x-4}=8^{2}$

$3x-4=2$

$3x=2+4$

$\frac{3x}{3}=\frac{6}{3}$

$x=2$

Your Turn:

Algebraically determine the solution for each of the following equations:

a) $3^{2x+1} = 3^{x+2}$

$2x + 1 = x + 2$

$2x - x = 2 - 1$

$\boxed{x = 1}$

b) $4^{3x+5} = 2^{4x-3}$

$2^{2(3x+5)} = 2^{4x-3}$

$2(3x+5) = 4x - 3$

$6x + 10 = 4x - 3$

$6x - 4x = -3 - 10$

$2x = -13$

$x = -\frac{13}{2}$ or $-6\frac{1}{2}$

c) $3(2)^{3x-2} = 48$

d) $9(2^{3x+5}) - 8 = 28$

Example 5:

Solve each equation algebraically:

Fraction in the base = Negative Exponent

a) $5^x = \frac{1}{125}$

$5^x = \frac{1}{5^3}$

$5^x = 5^{-3}$

$\boxed{x = -3}$

b) $(32)^{x-2} = \left(\frac{1}{4}\right)^{5x-3}$

$32^{x-2} = 4^{-(5x-3)}$

$2^{\boxed{5(x-2)}} = 2^{\boxed{-2(5x-3)}}$

$5(x-2) = -2(5x-3)$

$5x - 10 = -10x + 6$

$15x = 16$

$\boxed{x = \frac{16}{15}}$

Your Turn:

a) $\left(\frac{1}{8}\right)^{x-3} = 16^{2x+1}$

b) $2(4)^{2x} = \frac{1}{32}$

Example 6: $b^{\frac{m}{n}} = \sqrt[n]{b^m}$

Solve each equation algebraically:

Radical = Fractional Exponent

a) $\sqrt{8} = 2^{3x-4}$

$8^{\frac{1}{2}} = 2^{3x-4}$

$2^{3(\frac{1}{2})} = 2^{3x-4}$

$2^{\frac{3}{2}} = 2^{3x-4}$

$\frac{3}{2} = 3x-4$

$4 + \frac{3}{2} = 3x$

$\frac{11}{2} \div 3 = \frac{3x}{3}$

$\frac{11}{6} = x$

b) $5^{x+2} = \sqrt[3]{25}$

$5^{x+2} = \sqrt[3]{5^2}$

$5^{x+2} = 5^{\frac{2}{3}}$

$x+2 = \frac{2}{3}$

$x = \frac{2}{3} - 2$

$x = -\frac{4}{3}$

Your Turn:

a) $27^{2x-1} = \sqrt[3]{3^{2x}}$

b) $\sqrt{3^x} = 9^{2x+1}$

Example 7:

Solve each equation algebraically:

a) $9^{x-1} \times 81^{2x-1} = 27^{3x-2}$

$3^{2(x-1)} \times 3^{4(2x-1)} = 3^{3(3x-2)}$

$2(x-1)+4(2x-1)=3(3x-2)$

$2x-2+8x-4=9x-6$

$2x+8x-9x=-6+4+2$

$\boxed{x=0}$

b) $\dfrac{64^{x-1}}{16^{2x+2}} = 2^{x-2}$

$\dfrac{2^{6(x-1)}}{2^{4(2x+2)}} = 2^{x-2}$

$6(x-1)-4(2x+2)=x-2$

$6x-6-8x-8=x-2$

$6x-8x-x=-2+8+6$

$\dfrac{-3x}{-3}=\dfrac{12}{-3}$

$\boxed{x=-4}$

c) $5^{x^2+2x} = 125$

$5^{x^2+2x} = 5^{3}$

$x^2+2x=3$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$\boxed{x=-3}$ or $\boxed{x=1}$

Your Turn:

a) $4^{3x+2} \times 32^{x-2} = 8^{3x-4}$

b) $\dfrac{125^{2x+1}}{625^{x+2}} = 3125^{x+2}$

Example 8: Identify and correct the error.

$\sqrt{5} = 25^{3x+4}$

$5^{\frac{1}{2}} = 5^{2(3x+4)}$

$5^{\frac{1}{2}} = 5^{6x+\boxed{4}}$ 8

$\frac{1}{2} = 6x + 4$

$2 = 12x + 8$

$-6 = 12x$

$-\frac{1}{2} = x$

$5^{\frac{1}{2}} = 5^{6x+8}$

$\frac{1}{2} = 6x+8$

$\frac{1}{2} - 8 = 6x$

$\frac{1}{2} - \frac{16}{2} = 6x$

$\frac{-\frac{15}{2}}{6} = \frac{6x}{6}$

$-\frac{15}{12} = x$

Practice:

p. 361, #2abcd, 4cdef, 5abc, 7bdf

The half-life of a radioactive isotope is 30 hours. The amount of radioactive isotope A(t), at time t, can be modelled by the function:

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Algebraically determine how long it will take for a sample of 1792 mg to decay to 56 mg.

Solution:

$$\frac{56}{1792} = \frac{\cancel{1792}\left(\frac{1}{2}\right)^{\frac{t}{30}}}{\cancel{1792}}$$

$$\frac{1}{32} = \left(\frac{1}{2}\right)^{t/30}$$

$$\frac{1}{2^5} = \left(\frac{1}{2}\right)^{t/30}$$

$$\left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{t/30}$$

$$5 \cdot 30 = \frac{t}{30} \cdot 30$$

$$\boxed{60 = t}$$

→ 60 hours

The population of trout growing in a lake can be modeled

by the function $P(t) = 200(2)^{\frac{t}{5}}$ where $P(t)$ represents the number of trout and t represents the time in years after the initial count. How long will it take for there to be 6400 trout?

Note:

- the value of 200 represents the initial number of trout
- the number of trout doubles every 5 years

Solution:

$$\frac{6400}{200} = \frac{\cancel{200}(2)^{\frac{t}{5}}}{\cancel{200}}$$

$$32 = 2^{t/5}$$

$$2^{5} = 2^{t/5}$$

$$5 \cdot 5 = \frac{t}{5} \cdot 5$$

$$t = 25$$

The half life of Radon 222 is 92 hours. From an initial sample of 48g, how long would it take to decay to 6g?

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$